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Potential Theory over Local and Global Fields, I

DAVID HARBATER*

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

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INTRODUCTION

For a curve X over a local or global field K, Néron's pairing [Ne] is defined on $\mathrm{Div}_0(X)$, and extensions to all of $\mathrm{Div}(X)$ are given by Arakelov's Green's functions [Ar] if $K = \mathbf{C}$. This paper studies Green's pairings induced by a projective embedding and generalizes the notion of Green's pairings from $\mathbf{C}$ to more general fields K. Section 1 studies Green's functions on complex varieties under metrics induced by projective embeddings and focuses especially on curves, where there is an induced pairing. These Green's functions turn out to have quite explicit descriptions, relating to the Hopf bundle and the "chordal distance" on $\mathbf{P}^n_{\mathbf{C}}$. Section 2 then generalizes this potential theory to arbitrary local or global fields K and constructs "Green's pairings" on curves via the "K-Hopf bundle." This provides a new construction of Néron's pairing on curves and (in the complex case) of Arakelov's pairings under projective metrics. Again, a key ingredient is the "chordal distance" on $\mathbf{P}^n$, which in the non-archimedean case can be interpreted in terms of the order of contact between curves on $\mathbf{P}^n$ over a base curve. This paper is the first of two concerning potential theory over local and global fields. The sequel will relate the potential theory on curves at non-archimedean places to the intersection theory on arithmetic surfaces and develop a functorial "abstract potential theory" satisfying an adjunction formula.

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1. POTENTIAL THEORY IN COMPLEX PROJECTIVE SPACE

This section studies Green's functions on subvarieties V of complex projective space, with respect to the Fubini–Study metric. We show that these functions, which are solutions to partial differential equations, can in fact

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be described algebraically. The description has an interpretation in terms of the Hopf bundle on V arising from the projective embedding, and in the case of hyperplane sections the Green's functions may be given in terms of an explicit distance function which induces the Fubini–Study metric (but which is not the geodesic distance).

In Section 1.1 we discuss this distance function on projective space and its relation to Fubini–Study. Section 1.2 uses this discussion to analyze Green's functions associated to complete intersection divisors (i.e., hypersurface sections) on subvarieties V of $\mathbf{P}^n$. Section 1.3 discusses the case of interest in Arakelov theory, viz. curves. In this case Green's functions give rise to a symmetric "Green's pairing" between arbitrary divisors having disjoint support, and we show how Green's pairings with respect to Fubini–Study may be described rather explicitly and that in some situations they take on log algebraic values.

The results of this section will motivate the construction of Green's functions and Green's pairings over arbitrary local and global fields in Section 2.

1.1. *Fubini–Study Metric and the Chordal Distance Function*

The standard Hermitian metric on $\mathbf{P}^n(\mathbf{C})$ is the Fubini–Study metric, which may be defined as follows. Give $\mathbf{C}^{n+1}$ the standard Euclidean metric $ds^2 = \Sigma\, dz_j \otimes d\bar{z}_j$, whose associated $(1, 1)$-form $\omega = -\frac{1}{2}\operatorname{Im} ds^2$ is given by $\frac{i}{2}\sum dz_j \wedge d\bar{z}_j$. Then the $(1, 1)$-form $dd^c \log |z|^2 = \frac{i}{2}\partial\bar{\partial}\log |z|^2$ descends to a well-defined differential form on $\mathbf{P}^n(\mathbf{C})$ [GH, p. 30], and the Fubini–Study metric is the Hermitian metric on $\mathbf{P}^n(\mathbf{C})$ for which this is the associated $(1, 1)$-form. Since the above construction is invariant under the action of the unitary group $U(n+1)$, and since the above $(1, 1)$-form at the point $O = (1 : 0 : \cdots : 0) \in \mathbf{P}^n(\mathbf{C})$ is $\omega = \frac{i}{2}\sum dw_j \wedge d\bar{w}_j$ (where $w_0, ..., w_n$ are homogeneous coordinates on $\mathbf{P}^n$), the Fubini–Study metric can also be characterized as the unique unitarily invariant metric on $\mathbf{P}^n(\mathbf{C})$ whose $(1, 1)$-form at O is ω. Alternatively, consider the Hopf fibration $\pi\colon S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$, where S^{2n+1} is regarded as the unit sphere in $\mathbf{C}^{n+1}$, and the circle bundle π is the restriction to S^{2n+1} of the $\mathbf{C}^*$-bundle $\mathbf{C}^{n+1} - \{0\} \to \mathbf{P}^n(C)$. Restricting the Riemannian metric on $\mathbf{C}^{n+1}$ to S^{2n+1} yields a metric on S^{2n+1} which then descends to a Riemannian metric on $\mathbf{P}^n(\mathbf{C})$ (cf. [GH, p. 465]). This latter metric is unitarily invariant and is given by the form ω at the point O, and hence agrees with the Fubini–Study metric as described above.

Yet another method of obtaining the Fubini–Study metric is as follows: to each point P in $\mathbf{P}^n(\mathbf{C})$, let $M_P \in M_{n+1}(\mathbf{C})$ be the matrix corresponding to the linear transformation of $\mathbf{C}^{n+1}$ given by orthogonal projection onto the complex line $L_P \subset \mathbf{C}^{n+1}$ corresponding to P. Let $\mathbf{P}^n(\mathbf{C}) \hookrightarrow M_{n+1}(\mathbf{C})$ be the C^∞-embedding given by $P \mapsto M_P$. (Here we identify $M_{n+1}(\mathbf{C})$ with the

Hermitian inner product space $\mathbf{C}^N$, where $N=(n+1)^2$, so that $|M|^2=\operatorname{tr}(M^*M)$ for $M\in M_{n+1}(\mathbf{C})$.) Then as shown below (Proposition 1.1.3), the standard Riemannian metric on $M^{n+1}(\mathbf{C})$ induces the Riemannian metric on $\mathbf{P}^n(\mathbf{C})$ associated to Fubini–Study.

First, for A, $B\in\mathbf{C}^{n+1}-\{0\}$, define the cosine of the complex angle between (the complex lines connecting the origin to) A and B to be $\cos(A, B)=A\cdot B/|A|\,|B|$, where $\cdot$ is the standard Hermitian inner product. If A, B respectively lie over P, $Q\in\mathbf{P}^n(\mathbf{C})$, define $|\cos(P, Q)|=|\cos(A, B)|$. Note that this is well defined and that $0<|\cos(P, Q)|<1$. Also, write $|\sin(P, Q)|=(1-|\cos(P, Q)|^2)^{1/2}$. Finally, for any $P\in\mathbf{P}^n(\mathbf{C})$, say that a choice of homogeneous coordinates $(a_0:\cdots:a_n)$ is *normalized* if $|A|=1$, where $A=(a_0, \ldots a_n)\in\mathbf{C}^{n+1}$. Thus A is a point over P on the Hopf bundle.

1.1.1. LEMMA. (a) *If* $P\in\mathbf{P}^n(\mathbf{C})$, *with normalized homogeneous coordinates* $(a_0:\cdots:a_n)$, *then* M_P *is the matrix* $(a_\alpha\bar{a}_\beta)_{\alpha,\beta}$.

(b) *Let* $\mathcal{M}=\{M\in M_{n+1}(\mathbf{C})\mid M=M^*,\ M^2=M,\ \operatorname{rk} M=1\}$, *the group of rank* 1 *Hermitian idempotents. Then every element of* $\mathcal{M}$ *has norm* 1 *and trace* 1, *and* $M_P\in\mathcal{M}$ *for all* $P\in\mathbf{P}^n(\mathbf{C})$.

(c) *For* P, $Q\in\mathbf{P}^n(\mathbf{C})$, $\operatorname{tr}(M_PM_Q)=|\cos(P, Q)|^2$.

Proof. (a) For P as above, let N_P be the matrix $(a_\alpha\bar{a}_\beta)_{\alpha,\beta}$. Note that the ith entry in $N_P\cdot(a_0, \ldots, a_n)^t$ is $\sum_\beta a_\alpha\bar{a}_\beta a_\beta=a_\alpha$. So N_P fixes $(a_0, \ldots, a_n)$, and since N_P has rank 1 it follows that N_P is a projection onto the complex line corresponding to P. To show that $N_P=M_P$, it suffices to show that the image of (N_P-1) is orthogonal to $(a_0, \ldots, a_n)$. Now N_P is Hermitian, since $(N_P^*)_{\alpha\beta}=(\bar{N}_P)_{\beta\alpha}=(\bar{a}_\beta a_\alpha)=(N_P)_{\alpha\beta}$. Thus N_P-1 is also Hermitian, and so $(a_0, \ldots, a_n)\in\ker(N_P-1)=\ker(N_P-1)^*=(\operatorname{im}(N_P-1))^\perp$. Thus $N_P=M_P$, as asserted.

(b) If $M\in\mathcal{M}$ then $|M|=1$ because M is a non-zero idempotent and $|AB|=|A|\cdot|B|$ for A, $B\in M_n(\mathbf{C})$. Thus $\operatorname{tr} M=1$ for $M\in\mathcal{M}$, since $\operatorname{tr} M=\operatorname{tr}(M^2)=\operatorname{tr}(M^*M)=|M|^2=1$. Finally, since M_P is a rank 1 orthogonal projection, it is immediate that $M_P\in\mathcal{M}$.

(c) With respect to normalized coordinates $P=(a_0:\cdots:a_n)$ and $Q=(b_0:\cdots:b_n)$,

$$\begin{aligned}\operatorname{tr}(M_PM_Q)&=\sum_\alpha (M_PM_Q)_{\alpha\alpha}=\sum_\alpha\sum_\beta (M_P)_{\alpha\beta}(M_Q)_{\beta\alpha}\\&=\Big(\sum_{\alpha,\beta} a_\alpha\bar{a}_\beta\Big)\Big(\sum_{\alpha,\beta} b_\beta\bar{b}_\alpha\Big)\\&=\Big(\sum_\alpha a_\alpha\bar{b}_\alpha\Big)\Big(\sum_\beta b_\beta\bar{a}_\beta\Big)\\&=(A\cdot B)(B\cdot A)\\&=|\cos(P, Q)|^2.\quad\blacksquare\end{aligned}$$

On $\mathbf{P}^n(\mathbf{C})$, define the *chordal distance function* by $\rho(P, Q) = \sum_{i<j} |a_i b_j - a_j b_i|$, if $P = (a_0 : \cdots : a_n)$ and $Q = (b_0 : \cdots : b_n)$, in normalized coordinates. (Cf. also [Ru, Section 1.1].) This terminology is justified by the following result, which says that up to a constant, this distance is the length of the chord joining the images of P and Q in $M_{n+1}(\mathbf{C})$.

1.1.2. PROPOSITION. *Let $P, Q \in \mathbf{P}^n(\mathbf{C})$. Then $|M_P - M_Q|^2 = 2\,|\sin(P, Q)|^2 = 2\rho(P, Q)^2$, and so $|M_P - M_Q| = \sqrt{2}\rho(P, Q)$.*

Proof. Choose normalized coordinates $(a_0 : \cdots : a_n)$ and $(b_0 : \cdots : b_n)$ for P and Q, respectively. Then

$$\begin{aligned}
|M_P - M_Q|^2 &= \operatorname{tr}(M_P - M_Q)^* \,(M_P - M_Q) \\
&= \operatorname{tr}(M_P + M_Q - M_P M_Q - M_Q M_P) \\
&= 2 - 2\operatorname{tr}(M_P M_Q) \\
&= 2 - 2\,|\cos(P, Q)|^2 \\
&= 2\,|\sin(P, Q)|^2.
\end{aligned}$$

And since the coordinates have been normalized,

$$\begin{aligned}
|\sin(P, Q)|^2 &= 1 - |\cos(P, Q)|^2 \\
&= \Big(\sum_i |a_i|^2\Big)\Big(\sum_j |b_j|^2\Big) - \Big(\sum_i a_i \bar{b}_i\Big)\Big(\sum_j \bar{a}_j b_j\Big) \\
&= \Big(\sum_{i,j} |a_i|^2\,|b_j|^2\Big) - \Big(\sum_i |a_i|^2\,|b_i|^2\Big) - \Big(\sum_{i \neq j} a_i \bar{a}_j \bar{b}_i b_j\Big) \\
&= \Big(\sum_{i \neq j} |a_i|^2\,|b_j|^2\Big) - \Big(\sum_{i \neq j} a_i \bar{a}_j \bar{b}_i b_j\Big) \\
&= \sum_{i<j} (a_i b_j - a_j b_i)(\bar{a}_i \bar{b}_j - \bar{a}_j \bar{b}_i\Big) \\
&= \sum_{i<j} |a_i b_j - a_j b_i|^2 \\
&= \rho(P, Q)^2.
\end{aligned}$$

So $|M_P - M_Q| = \sqrt{2}\rho(P, Q)$. ∎

1.1.3. PROPOSITION. *Consider the Riemannian metric on $\mathbf{P}^n(\mathbf{C})$ obtained by pulling back the standard Riemannian metric on $\mathbf{C}^N$, via the embedding $\mathbf{P}^n(\mathbf{C}) \subsetneq \mathbf{C}^N$. Then the associated Hermitian metric on $\mathbf{P}^n(\mathbf{C})$ is equal to twice the Fubini–Study metric.*

Proof. A Hermitian metric ds^2 is determined by the corresponding Riemannian metric $\operatorname{Re} ds^2$ [GH, p. 28], and that metric on the Euclidean space $M_{n+1}(\mathbf{C})$ is $\sum (dx_{\alpha\beta} \otimes dx_{\alpha\beta} + dy_{\alpha\beta} \otimes dy_{\alpha\beta})$, where $z_{\alpha\beta} = x_{\alpha\beta} + iy_{\alpha\beta}$. Now there is a unique pullback of this Riemannian metric to $\mathbf{P}^n(\mathbf{C})$ which preserves inner products [BC, p. 132, item 3], viz. $2\sum (dx_j \otimes dx_j + dy_j \otimes dy_j)$, where $z_j = x_j + iy_j$. The corresponding Hermitian metric has associated (1, 1)-form $i \sum dz_j \wedge d\bar{z}_j$, which is twice that of the Fubini–Study metric. ▮

From this we obtain yet another construction of Fubini–Study:

1.1.4. COROLLARY. *The chordal distance function on* $\mathbf{P}^n(C)$ *induces the Fubini–Study metric.*

Proof. Immediate from 1.1.2 and 1.1.3. ▮

Note that the chordal distance function is *not* the same as the geodesic distance function induced by Fubini–Study; they only agree infinitesimally. On the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, the chordal distance between P and Q is equal to the length of the chord in $\mathbf{R}^3$ connecting P and Q, if $\mathbf{P}^1(\mathbf{C})$ is embedded as a round sphere in $\mathbf{R}^3$ of diameter 1. In general, there is also the following interpretation:

1.1.5. PROPOSITION. (a) *Let* $P, Q \in \mathbf{P}^n(\mathbf{C})$, *let* $A \in S^{2n+1}$ *lie over* P, *and let* L_Q *be the complex line in* $\mathbf{C}^{n+1}$ *corresponding to* Q. *Then* $\rho(P, Q)$ *is equal to the distance from* A *to* L_Q *in* $\mathbf{C}^{n+1}$.

(b) *Thus* $\mathbf{P}^n$ *has diameter* 1, *relative to* ρ.

Proof. (a) By transitivity of the isometry group, we may assume $Q = (1 : 0 : \cdots : 0)$. Let $P = (a_0 : \cdots : a_n)$ and $A = (a_0, ..., a_n)$, so that the orthogonal projection of A onto L_Q is $A' = (a_0, 0, ..., 0)$. Then the distance from A to L_Q is $|A - A'| = |(0, a_1, ..., a_n)| = \rho(P, Q)$.

(b) For any $P, Q \in \mathbf{P}^n(\mathbf{C})$, the origin O lies on L_Q, and so by part (a) we have $\rho(P, Q) \leqslant |A - O| = 1$. Also, if $P = (1 : 0 : \cdots : 0)$ and $Q = (0 : \cdots : 0 : 1)$, then $\rho(P, Q) = 1$. So the diameter equals 1. ▮

More generally, we have

1.1.6. COROLLARY. *Let* $X \subset \mathbf{C}^{n+1}$ *be a vector subspace and let* $A \in S^{2n+1}$, *and let* $H \subset \mathbf{P}^n(\mathbf{C})$ *and* $P \in \mathbf{P}^n(\mathbf{C})$ *be their images in projective space. Then the distance from* P *to* H *in* $\mathbf{P}^n(\mathbf{C})$ (*i.e.*, $\min\{\rho(P, Q) \mid Q \in H\}$) *is equal to the distance from* A *to* X *in* $\mathbf{C}^{n+1}$.

Proof. Immediate from 1.1.5(a), since the distance from A to X is equal to the minimum distance from A to L_Q, where Q ranges over the points of H. ▮

If F is a homogeneous polynomial in $z_0, ..., z_n$, and $A \in S^{2n+1} \subset \mathbf{C}^{n+1}$ lies above $P \in \mathbf{P}^n(\mathbf{C})$, we define $|F|\,(P) = |F(A)|$. Given P, this is independent of the choice of A. Also, if F has degree d and $P = (a_0 : \cdots : a_n)$, then $|F|\,(P) = |F(a_0, ..., a_n)|/|(a_0, ..., a_n)|^d$.

1.1.7. Proposition. *Let H be a complex hyperplane in $\mathbf{P}^n(\mathbf{C})$, defined by the vanishing of a linear form $F = \sum_{j=0}^n \alpha_j z_j$, normalized so that $|(\alpha_0, ..., \alpha_n)| = 1$. Then for $P \in \mathbf{P}^n(\mathbf{C})$, the distance from P to H is equal to $|F|\,(P)$.*

Proof. After composing with an isometry, we may assume that $F = z_0$. Write $P = (a_0 : \cdots : a_n)$ in normalized coordinates, and let $A = (a_0, ..., a_n)$. By 1.1.6, the distance from P to H is equal to the distance from A to $X : z_0 = 0$ in $\mathbf{C}^{n+1}$. The closest point on X to A is $A' = (0, a_1, ..., a_n)$, and so the latter distance is $|A - A'| = |a_0| = |F|\,(P)$, as claimed. ∎

1.2. *Green's Functions in Complex Projective Space*

Let M be a complex manifold of dimension n. Recall that a (p, p) current on M is a linear functional on the space $A_c^{n-p,n-p}(M)$ of compactly supported $(n-p, n-p)$ forms. Note that any compactly supported (p, p) form ω on M can be regarded as a (p, p) current, via

$$\phi \mapsto \int_M \omega \wedge \phi.$$

Given a submanifold $N \subset M$ of codimension p, there is an induced (p, p) current T_N. Namely, T_N is the linear functional on $A_c^{n-p,n-p}(M)$ given by

$$\phi \mapsto \int_N \phi.$$

Extending by linearity, we obtain for any codimension p cycle Z an induced (p, p)-current T_Z.

Note that if M is compact and N is as above, then by Poincaré duality there is a (p, p) form ω_N which is dual to N, i.e., such that for all closed $(n-p, n-p)$ forms ϕ,

$$\int_N \phi = \int_M \omega \wedge \phi.$$

Thus $\omega = T_N$ as a (p, p) current.

Now suppose that M is compact and that ω is the $(1, 1)$ form on M corresponding to a Kahler metric. Then we will define a *Green's function* associated to a divisor H on the hermitian manifold (M, ω) to be a function g on M satisfying the identity of currents

$$(1/\pi i)\, \partial\bar{\partial} g = T_H - r\omega \tag{*}$$

for some real number r.

If H is a divisor such that $T_H - r\omega$ is homologous to zero for some $r \in \mathbf{R}$ (as is the case for all H if $\dim M = 1$), then an associated Green's function may be constructed by means of the Green's operator, as in [GH, p. 169 or GS, Section 1.2]. Also, by the Poincaré–Lelong equation [GH, p. 388], g is a Green's function for H if and only if

(i) for every open set $U \subset M$ and local equation $f \in O(U)$ for H, $g + \log |f|$ extends to a C^∞-function on U; and

(ii) g satisfies the equation $\partial\bar{\partial} g = -r\omega$ as functions on $M - H$, for some real number r.

If g is a Green's function for H on (M, ω), then by (i) the function $G = \exp(-g)$ extends to a C^∞-function on M (vanishing on H). We call the extension of G to a function on M a *multiplicative Green's function* associated to H on (M, ω).

1.2.1. PROPOSITION. *Let (M, ω) be a compact Kahler manifold of dimension n, and g a Green's function for a hypersurface H. Then g is unique up to adding a constant, and the value of r in (*) is*

$$\frac{1}{n} \frac{\text{volume } H}{\text{volume } M}.$$

Proof. The first assertion follows from the second, for if g, g^* are two Green's functions, then (by (i)) $g - g^*$ extends to a C^∞-function on M whose Laplacian is zero; since M is compact, this forces $g - g^*$ to be constant.

To prove the second assertion, observe that (*) says that for any $\phi \in A^{n-1,n-1}(M)$,

$$\int_M (1/\pi i)(\partial\bar{\partial} g) \wedge \phi = \int_H \phi - \int_M r\omega \wedge \phi.$$

The integrand on the left-hand side is equal to $(1/\pi i)\, \partial\bar{\partial} g \wedge \phi$, because the metric is Kahler; so the left-hand side equals 0, by Stokes' theorem. In particular, setting $\phi = \omega^{n-1}$ (the $(n-1)$th exterior power of ω) and using the fact that the volume forms on M and H are $(1/n!)\, \omega^n$ and $(1/(n-1)!)\, \omega^{n-1}$, respectively (by the Wirtinger theorem [GH, p. 31]), we obtain $(n-1)!$ (vol H) $= r\cdot n!$ (vol M). Thus $r=(1/n)(\mathrm{vol}\,H)/(\mathrm{vol}\,M)$, as claimed. ∎

1.2.2. LEMMA. *Let $f\colon N \hookrightarrow M$ be a holomorphic embedding of complex manifolds and let H be a hypersurface in M such that $f^{-1}(H)$ is a hypersurface in N. Let g be a Green's function for H on M, relative to some Kahler metric on M. Then $f^*g = g\circ f$ is a Green's function for $f^{-1}(H)$ on N, relative to the induced metric on N.*

Proof. Say $\dim M = m$ and $\dim N = n$. Let ω be the $(1, 1)$ form on M corresponding to the hermitian metric. Thus the pullback of ω to N is the $(1, 1)$ form corresponding to its hermitian metric [GH, p. 29].

We are given that for any $\phi\in A^{m-1,m-1}(M)$,

$$\int_M (1/\pi i)(\partial\bar\partial g)\wedge\phi = \int_H \phi - \int_M r\omega\wedge\phi \tag{1}$$

for appropriate r, and we want to show that for any $\eta\in A^{n-1,n-1}(N)$,

$$\int_N (1/\pi i)(\partial\bar\partial g)\wedge\eta = \int_{H\cap N} \eta - \int_N r\omega\wedge\eta. \tag{2}$$

Now, as above, by Poincaré duality there is an $(m-n, m-n)$ form γ and a $(1, 1)$ form α whose induced currents are T_N and T_H, respectively. That is, they are dual to N and H, respectively—i.e., for all closed (n, n) forms σ and all closed $(m-1, m-1)$ forms τ,

$$\int_N \sigma = \int_M \gamma\wedge\sigma \qquad\text{and}\qquad \int_H \tau = \int_M \alpha\wedge\tau.$$

And since H and N intersect properly, $\alpha\wedge\gamma$ is the $(m-n+1, m-n+1)$ form whose induced current is $T_{H\cap N}$.

So given an $\eta\in A^{n-1,n-1}(N)$, let $\phi=\gamma\wedge\eta$. Thus $\int_H\phi=\int_M\alpha\wedge\phi=\int_{H\cap N}\eta$. Similarly, the left-hand sides of (1) and (2) are equal, as are the second terms on the right-hand sides of (1) and (2). Thus (2) follows from (1). ∎

1.2.3. PROPOSITION. *Let N be a complete non-singular variety together with a choice of Kahler metric ω. For each Cartier divisor D on N such that T_D is homologous to a real multiple of ω, let $g_D = g_{D,N,\omega}$ be a Green's function for D on (N,ω), so that g_D is determined up to a constant. This assignment satisfies the following properties*:

(a) *g_D is a* Weil function *for D; i.e., if P is a point on the support of D and f is a rational function on N which is a local equation for D at P, then*

$g_D + \log|f|$ extends to a locally bounded continuous function on an affine neighborhood of P.

(b) *Given (N, ω) and two such divisors C and D on N, we have $g_{C+D} = g_C + g_D$, modulo constant functions.*

(c) *If D is numerically equivalent to zero, then g_D depends only on D and N, and not on the hermitian metric.*

(d) *If $D = (f)$ is linearly equivalent to zero, then g_D is given by $g_D(P) = -\log|f(P)|$, modulo constant functions.*

(e) *Let $\mu: N \to M$ be a morphism of projective varieties, and let D be a divisor on M which is algebraically equivalent to zero, such that $C = \mu^{-1}(D)$ is a divisor on N. Then $g_{C,N} = g_{D,M} \circ \mu$ modulo constant functions (where the Green's functions are with respect to any choices of metric).*

Proof. (b) Immediate from the definition of Green's function.

(c) Since D is numerically equivalent to zero, by Proposition 1.2.1, $g_{D,M,\omega}$ satisfies the differential equation $(1/\pi i)\,\partial\bar{\partial}g = T_D$ on M, and this is independent of the embedding.

(d) Write $D = D' - D''$, where D' and D'' are the divisors of zeroes and of poles of F. Applying the Poincaré–Lelong equation [GH, p. 388] to f and f^{-1} on $(N - \operatorname{supp} D'')$ and $(N - \operatorname{supp} D')$, we find that $(i/\pi)\,\partial\bar{\partial}\log|f| = T_D$. But by the proof of (c) above, this equals $(1/\pi i)\,\partial\bar{\partial}g_D$. So (d) follows.

(a) The divisor $C = (f)$ on N has Green's function $g_C = -\log|f|$ by part (d), and P is not on the support of $D - C$. Thus $g_D + \log|f| = g_D - g_C = g_{D-C}$, which is C^∞ in a neighborhood of P.

(e) Since D and hence C are algebraically equivalent to zero, by (c), the Green's functions $g_{D,M}$ and $g_{C,N}$ are independent of the choice of metric, and respectively satisfy the differential equation $(1/\pi i)\,\partial\bar{\partial}g = T_D$ on M, and $(1/\pi i)\,\partial\bar{\partial}g = TC$ on N. Pulling back the first of these equations to N shows that $g_{D,M} \circ \mu$ also satisfies the latter, and hence this pullback agrees with $g_{C,N}$ modulo constants (the difference being harmonic). ∎

Note in particular that for divisors D which are algebraically equivalent to zero on some projective variety N, the function $g_{N,D}$ (which is well defined, independent of any metric or embedding by (c)) is a *Néron function* for D on N—i.e., the assignment $D \mapsto g_{N,D}$ satisfies the above properties (a), (b), (d), (e). Moreover, the Néron function is unique up to constants, by [La, Chap. 11, Theorem 3.1].

Note also that, while Green's functions are only determined up to adding a constant, they can be normalized so that the above properties (a)–(e) hold, by using the normalization such that $\int_N g\,d\mu = 0$.

We now turn our attention to the case of Green's functions in projective space:

1.2.4. PROPOSITION. *Let F be a homogeneous form defining a hypersurface H in $\mathbf{P}^n$. Then a multiplicative Green's function for H, relative to the Fubini–Study metric, is given by $G_H(P) = |F|\,(P)$.*

Proof. We want to show that if $P = (a_0 : \cdots : a_n)$ in coordinates which are not necessarily normalized and F is of degree r, then $G_H(P) = |F(a_0, ..., a_n)|/[\sum |a_i|^2]^{r/2}$ is a multiplicative Green's function. Namely, on the open set U_j we have $(i/\pi)\,\partial\bar\partial \log G_H(P) = (i/\pi)\,\partial\bar\partial \log |F(w_0, ..., w_n)| - (ri/\pi)\,\partial\bar\partial \log[\sum |w_i|^2]^{1/2}$. Here, the first term is equal to T_H, by the Poincaré–Lelong equation [GH, p. 388], applied to U_j. The second term is equal to $(r/\pi)\omega$, where ω is the $(1, 1)$-form corresponding to Fubini–Study (cf. Section 1.1). So $(i/\pi)\,\partial\bar\partial \log G_H = T_H - (r/\pi)\omega$, as claimed.

Note that the value of r in the above proof, combined with Proposition 1.1.1, implies the well-known fact that the volume of a hypersurface in $\mathbf{P}^n$ is proportional to its degree, viz.,

$$\mathrm{vol}\, H = n(\deg H)(\mathrm{vol}\, \mathbf{P}^n).$$

1.2.5. COROLLARY. *If H is a hyperplane in $\mathbf{P}^n$, the function*

$$G(P) = \textit{distance from } P \textit{ to } H$$

is a multiplicative Green's function on $\mathbf{P}^n$ for H.

Proof. By 1.2.4 and 1.1.7. ∎

By Lemma 1.2.2, the Fubini–Study metric and Green's functions on $\mathbf{P}^n(\mathbf{C})$ induce a Hermitian metric and associated Green's functions on subvarieties. Here, two subvarieties are isometric if and only if they are unitarily equivalent—i.e., there is an isometry of $\mathbf{P}^n(\mathbf{C})$ (given by a unitary transformation) taking one to the other.

If D is a hypersurface on a subvariety N of $\mathbf{P}^n$, call D a *complete intersection hypersurface* on N (relative to the given embedding of N in $\mathbf{P}^n$) if $D = H \cap N$ for some hypersurface H in $\mathbf{P}^n$. Equivalently, D is a complete intersection hypersurface if and only if there is a homogeneous form F such that D is defined by the vanishing of F on N. Let $\mathrm{CDiv}(N)$ be the group of divisors on N (the *complete intersection divisors*) generated by the complete intersection hypersurfaces relative to the given embedding in projective space.

1.2.6. THEOREM. *Say that D is a complete intersection hypersurface on $N \subset \mathbf{P}^n$, with D defined on N by the homogeneous form F. Then $G(P) = |F|\,(P)$ is a multiplicative Green's function for D on N (relative to the metric on N induced from $\mathbf{P}^n(\mathbf{C})$).*

Proof. By 1.2.2 and 1.2.4. ∎

Note that if D is a divisor on $N \subset \mathbf{P}^n$ such that some *multiple* rD ($r \in \mathbf{Z}$, $r \neq 0$) is a complete intersection hypersurface, then Theorem 1.2.6 gives a formula for a multiplicative Green's function for D on N, viz. $G(P) = |F(P)|^{1/r}$. More generally, define the group of $\mathbf{Q}$-divisors $\mathbf{Q}\mathrm{Div}(N)$ to be $\mathrm{Div}(N) \otimes_{\mathbf{Z}} \mathbf{Q}$ and define the group of complete intersection $\mathbf{Q}$-divisors $\mathbf{Q}\mathrm{CDiv}(N)$ to be $\mathrm{CDiv}(N) \otimes_{\mathbf{Z}} \mathbf{Q}$. Thus 1.2.6 gives a formula for a multiplicative Green's function on D for all $D \in \mathbf{Q}\mathrm{CDiv}(N)$.

Observe that by 1.2.6, multiplicative Green's functions for complete intersection hypersurfaces H on $N \subset \mathbf{P}^n$ actually arise from complex valued functions on the Hopf bundle. Namely, if H is defined by a form F, then G_H is the (descent to N of) the absolute value of the function F on the total space of the Hopf fibration.

1.3. *Green's Pairings on Complex Projective Curves*

Let X be a smooth complex curve, with a given Hermitian metric (which is necessarily Kahler). Then Green's functions exist for all divisors on X, and Arakelov showed that the Green's functions may be normalized so as to give a symmetric function on pairs of distinct points of X:

1.3.1. PROPOSITION [Ar]. *Let X be a smooth complete complex curve with Hermitian metric $d\mu$. Then there is a continuous map $g: X(\mathbf{C}) \times X(\mathbf{C}) - \Delta \to \mathbf{R}$ such that $g(P, Q) = g(Q, P)$ and such that for every $P \in X(\mathbf{C})$, the map $Q \mapsto g(P, Q)$ is a Green's function for P on X, relative to $d\mu$. Moreover, given $(X, d\mu)$, the pairing g is unique up to adding a constant.*

Indeed, Arakelov showed that if $c \in \mathbf{R}$, and if for each point $P \in X(\mathbf{C})$ the Green's function g_P for P is normalized by demanding that

$$\frac{1}{\operatorname{vol} X} \int_X g_P(Q)\, d\mu(Q) = c,$$

then we may take $g(P, Q) = g_P(Q)$. Moreover, his proof shows that each such g arises in this fashion and that any two such pairings (coming from different choices of constants, say c and c') differ by a constant (viz. $c' - c$).

We will call g a *Green's pairing* on X, relative to the metric $d\mu$, and we will call $G = \exp(-g)$ a *multiplicative Green's pairing.*

Remarks. (a) Arakelov chose to normalize the volume form on X by setting $\operatorname{vol} X = 1$ and to normalize g by taking the above constant $c = 0$. But these particular choices are unnecessary, and for $X \subset \mathbf{P}^n$ the natural volume is not 1, but rather it is proportional to the degree of X. So here we prefer to allow more arbitrary choices for these constants.

(b) If g and g' are Green's pairings for metrics $d\mu$ and $d\mu'$, then according to Arakelov [Ar, Proposition 3.2] there is a continuous function $\tau: X(\mathbf{C}) \to \mathbf{R}$ such that

$$g'(P, Q) = g(P, Q) + \tau(P) + \tau(Q). \tag{*}$$

Also, his proof shows that if g is a Green's pairing with respect to $d\mu_0$ and $\tau: X(\mathbf{C}) \to \mathbf{R}$ is continuous, then the bilinear pairing g' given by (*) is a Green's pairing with respect to another metric, viz. $d\mu_1 = d\mu_0 - (1/\pi i)\,\partial\bar{\partial}f$. Arakelov showed his result [Ar, Proposition 3.2] subject to his choice of normalization constants, but it easily carries over in general.

Using bilinearity, we may extend g to a function on pairs (D_1, D_2) of $\mathbf{Q}$-divisors on X having disjoint support. This pairing is again symmetric, and for any divisor D the function $g_D: P \mapsto g(D, P)$ is clearly a Green's function for D on X under the given metric (cf. Section 1.2 above). Similarly, writing $G = \exp(-g)$, we have that $G_D = \exp(-g_D)$ is a multiplicative Green's function for D on X.

Recall [La, Chap. 11, Theorem 3.6] that on a complete non-singular curve X there is a unique *Néron height pairing*—i.e., a bilinear symmetric pairing on divisors of degree 0 having disjoint supports, such that if $D = (f)$ is principal then $\langle D, C\rangle = -\log|f(C)|$ (where $f(C) = \prod f(P_i)/\prod f(Q_j)$ if $C = \sum P_i - \sum Q_j$), and such that $P \mapsto \langle D, Q - P\rangle$ is a locally bounded continuous function on $X - \operatorname{supp} D$ for all divisors D of degree 0 and all points Q. This is related to the *Néron pairing* $\langle\ ,\ \rangle$ on an abelian variety A, which is the unique translation invariant bilinear pairing $\operatorname{Div}(A) \times Z_0 X - \Delta \to \mathbf{R}$ (on pairs of divisors and zero cycles of degree 0 having disjoint support) such that $\langle (f), Z\rangle = -\log|f|(Z)$ for rational functions f and such that $P \to \langle D, P - P_0\rangle$ is locally bounded on $A(\mathbf{C}) - (\operatorname{supp} D)$, for fixed $D \in \operatorname{Div}(A)$ and $P_0 \in X(\mathbf{C}) - (\operatorname{supp} D)$ [Ne, Chap. II, Theorem 1; La, Chap. 11, Theorem 2.2]. Namely, let A be the Jacobian of a curve X, and let $\alpha: X \hookrightarrow A$ be a choice of Albanese map. If $D \in \operatorname{Div}(A)$ is algebraically equivalent to zero and $\alpha^*(D)$ has support disjoint from that of $Z \in Z_0 X$, then the Néron pairings on A and X are related by $\langle D, \alpha(Z)\rangle = \langle \alpha^*(D), Z\rangle$ [La, Chap. 11, proof of Theorem 3.6].

1.3.2. Corollary. *If X is a smooth complete complex curve, then the restriction of a Green's pairing g to divisors of degree 0 is independent of the choice of Hermitian metric and is independent of the choice of constant of normalization of g. Moreover, this restriction agrees with the Néron pairing on X.*

Proof. The independence of choice of metric follows from 1.2.3(c), and since $g_D(P - Q) = g_D(P) - g_D(Q)$, the restriction to pairs of degree 0 divisors is independent of the choice of normalizing constant for g_D. Since g is bilinear, symmetric, and continuous and since $g_{(f)}(P) = -\log|f(P)|$, these properties carry over to the restriction, and hence the uniqueness of the Néron pairing implies that these pairings must be equal. ∎

If X is a smooth complete complex curve, then there is a natural Hermitian inner product on the space $H^0(X, \Omega^1)$ of holomorphic differentials, given by

$$\langle \omega, \omega' \rangle = \int_X \omega \wedge \bar{\omega}'.$$

Let $\omega_1, ..., \omega_g$ be an orthonormal basis for $H^0(X, \Omega^1)$ (where $g = \text{genus } X$), and consider the $(1, 1)$-form $d\mu = (i/2g) \sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j$. This corresponds to the *canonical metric* on X, and there is an Arakelov-theoretic adjunction formula [Ar, Theorem 4.1; Ch, Proposition 4.1] and a Riemann–Roch formula [Fa, Theorem 3] with respect to the induced Green's pairing g, the *canonical Green's pairing* on X. Identifying $H^0(X, \Omega^1)$ with $H^0(A, \Omega^1)$, the canonical $(1, 1)$-form $d\mu$ on X corresponds to a $(1, 1)$-form on A, which we also denote by $d\mu$. Thus $d\mu$ is translation invariant on A, and is homologous to the current T_Θ for any theta-divisor Θ on A. So a divisor D on A has an associated Green's function relative to $d\mu$ if and only if D is homologous (or algebraically equivalent) to a real multiple of Θ. In the case that Θ generates homology, the condition on D is vacuous, and we have the following partial generalization of the equality given just before 1.3.2:

1.3.3. Proposition. *Let X be a smooth complete complex curve, and let g be the canonical Green's pairing on X. Let A be the Jacobian of X, let $\alpha: X \hookrightarrow A$ be the Albanese map, and let $\langle\, ,\, \rangle$ be the Néron pairing on A. If the Néron–Severi group* $\text{NS}(A) = \text{Div}(A)/\text{Div}_a(A)$ *has rank* 1 (*which is the case for generic X and for all elliptic curves*), *then*

$$\langle D, \alpha(Z) \rangle = g(\alpha^*(D), Z) \tag{*}$$

for all $D \in \text{Div}(A)$ such that $\alpha^* D$ has support disjoint from that of $Z \in Z_0 X$.

Proof. Since $\text{NS}(A)$ has rank 1, any $D \in \text{Div}(A)$ has a Green's function g_D on A, relative to $d\mu$. Here g_D is well defined up to a constant, and $g_D(Z)$ is uniquely determined if $Z \in Z_0 X$ has support disjoint from that of D. Moreover, the map $D \mapsto g_D$ (modulo constants) is translation invariant and satisfies the properties listed in 1.2.3. Thus the pairing

$$\text{Div}(A) \times Z_0(A) - \Delta \to \mathbf{R}, \qquad (D, Z) \mapsto g_D(Z)$$

between divisors and zero cycles of degree 0 haıving disjoint support must equal the Néron pairing on A.

Now by Lemma 1.2.2, $Q \mapsto g_D(\alpha(Q))$ is a Green's function for $\alpha^*(D)$ on X with respect to the canonical metric. Since $Q \mapsto g(\alpha^*(D), Q)$ is also such a Green's function, it follows that these must differ by a constant and thus agree as functions on $Z_0X - (\text{supp } D)$. So indeed, $\langle D, \alpha(Z)\rangle = g_D(\alpha(Z)) = g(\alpha^*(D), Z)$. ∎

Remarks. (a) The conclusion of 1.3.3 does not hold for arbitrary D without the assumption on the Néron–Severi group. Namely, according to Manin [Ma, pp. 336–337], there are elliptic curves V_i together with two-to-one branched covering maps $\phi_i\colon V_i \to \mathbf{P}^i$ $(i=1, 2)$, such that the following holds: Let $D_i = \phi_i^{-1}(\infty)$, let V be the normalization of $V_1 \times_{\mathbf{P}^1} V_2$ (so V is a curve of genus two), and let $p_i\colon V \to V_i$ be the natural projection maps. Then $p_1^{-1}(D_1)$ and $p_2^{-1}(D_2)$ are equal to the same divisor D on V, but the pullbacks to V of the Néron functions associated to D_1 and D_2 do not agree (even up to a constant). From this example of Manin, we construct the desired counterexample as follows: Let A be the Jacobian of V, and let $\alpha\colon V \to A$ be a choice of Albanese map. Then p_i induces a map $q_i\colon A \to \text{Jac}(V_i) = V_i$, and so $p_i = q_i \circ \alpha$. Let $D_i' = q_i^{-1}(D_i)$, so that D_1' and D_2' have the same pullback to V. Then the Néron function of D_i on V_i pulls back to the Néron function of D_i' on A, because of the functoriality of Néron functions with respect to homomorphisms of abelian varieties. Thus $\alpha^{-1}(D_i) = D$ for $i = 1, 2$, and yet the Néron functions of D_1 and D_2 have distinct pullbacks to V. So $\langle D_1, \alpha(Z)\rangle$ and $\langle D_2, \alpha(Z)\rangle$ cannot both be equal to $g(D, Z)$, for arbitrary D, Z as in 1.3.3. Thus the identity (*) cannot hold in general.

(b) The comments prior to 1.3.3 do suggest the possibility that if D is a theta divisor on A, then (*) might hold. Cf. also remarks following 2.3.10.

We now restrict to the case of $X \subset \mathbf{P}^n$ under the Fubini–Study metric and show how Theorem 1.2.6 can be used to obtain a rather explicit expression for the induced Green's pairing on X. Let $G = \exp(-g)$ be a multiplicative Green's pairing (which is unique up to a constant). Then we first describe G_D for D a $\mathbf{Q}$-complete intersection divisor and, afterwards (Proposition 1.3.9), do this for a more general $\mathbf{Q}$-divisor D.

1.3.4. Proposition. *Let X be a smooth curve in $\mathbf{P}^n$ and let $D \in \mathbf{Q}\text{CDiv}(X)$, say with $rD \in \text{Div}(X)$ defined by a homogeneous form F. Then there is a constant $k(F) > 0$ such that G_D is given by*

$$G_D(P) = k(F)(|F|\ (P))^{1/r}.$$

Proof. Immediate from the above, Theorem 1.2.6, and Proposition 1.2.1. ∎

The various constants $k(F)$ may be determined up to a single constant k (which depends upon the particular choice of g):

1.3.5. PROPOSITION. *Let $X \subset \mathbf{P}^n$ be a smooth curve, and let D_0 be a complete intersection divisor, say defined by the homogeneous form F_0. Then there is a positive constant k such that for all $D \in \mathbf{Q}\mathrm{CDiv}(X)$, if F is a homogeneous form defining the divisor rD, then*

$$k(F) = k^{(\deg D)} \, |F_0| \, (D)/|F| \, (D_0).$$

Proof. We claim that $k = k(F_0)$ suffices. To see this, write $D_0 = \sum n_i P_i$, and $D = \sum m_j Q_j$. Also write $n = \sum n_i$, $m = \sum m_j$. By symmetry, $g(D_0, D) = g(D, D_0)$, and so $k^m \prod |F_0| \, (Q_j)^{m_j} = k(F)^n \prod |F| \, (P_i)^{n_i}$. So the result follows. ∎

We wish to extend this result to the case of general D. So put the l_1-norm on the $\mathbf{Q}$-vector space $\mathbf{Q}\mathrm{Div}\, X$. That is, identify $\mathbf{Q}\mathrm{Div}(X)$ with the $\mathbf{Q}$-vector space spanned by the points of X and set $|\sum a_i P_i|_1 = \sum |a_i|$. Thus for an effective divisor $D \in \mathrm{Div}(X) \subset \mathbf{Q}\mathrm{Div}\ V$, $|D| = \deg D$. Then we have

1.3.6. LEMMA. $\mathbf{Q}\mathrm{CDiv}\ X$ *is l_1-dense in* $\mathbf{Q}\mathrm{Div}\ X$.

Proof. Let $D \in \mathbf{Q}\mathrm{Div}(X)$. We wish to show that D may be arbitrarily well approximated by elements of $\mathbf{Q}\mathrm{CDiv}(X)$. It suffices to show this in the case that D is an effective divisor of degree $s \geqslant 1$. Let g be the genus of X, let d be the degree of X in $\mathbf{P}^n$, and let H be a hyperplane section. Thus H defines a divisor of degree d on X. Then for $n > 0$, $D_n = ndgD - (n-1)\, sgH$ has degree $ndgs - (n-1)\, dgs = dgs \geqslant g$. Thus by the Riemann–Roch theorem, $H^0(D_n) \neq 0$, and the linear system $|D_n|$ is non-empty. Let E_n be an element of this linear system, i.e., an effective divisor linearly equivalent to D_n. Then $|E_n|_1 = \deg E_n = dgs$. Let $F_n = ndgD - E_n$. Thus F_n is linearly equivalent to $(n-1)\, sgH$, and hence lies in $\mathrm{CDiv}(X)$. Hence $D - (1/ndg)\, E_n$ lies in $\mathbf{Q}\mathrm{CDiv}(X)$, where $(1/ndg)\, E_n$ has l_1-norm $dgs/ndg = s/n$. Since n is arbitrary, this proves density. ∎

To use the above result, we need to extend by continuity. Unfortunately, $G_D(P)$ does not vary well with D, because of vanishing along the support of D. For example, for any point P, we have that $rP \to 0$ as $r \to 0$, but $G_{rP}(P) = 0$ which does not approach $G_0(P) = 1$. So instead we use an auxiliary function η_D, which does not vanish. Namely, if $D = \sum a_j Q_j$ is a $\mathbf{Q}$-divisor on a projective curve $X \subset \mathbf{P}^n_{\mathbf{C}}$, define the distance function ρ_D by $\rho_D(P) = \prod \rho(Q_j, P)^{a_j}$, and set $\eta_D = G_D/\rho_D \colon X(\mathbf{C}) - (\operatorname{supp} D) \to \mathbf{R}$.

1.3.7. LEMMA. (a) *For any divisor D, the function $\eta_D = G_D/\rho_D$ extends to a continuous positive valued function $X(\mathbf{C}) \to \mathbf{R}$.*

(b) *For $P, Q \in \mathbf{X}$, let $\eta(P, Q) = \eta_P(Q)$. Then $\eta: X \times X \to \mathbf{R}$ is a continuous symmetric positive-valued function, and there exists $k > 1$ such that for all $P, Q \in X$, $k^{-1} < \eta(P, Q) < k$.*

Proof. (a) By linearity, we may assume $D = P$, a single point. Let t be a local parameter on X at P. On a punctured neighborhood of P, we may write $\eta_P(Q) = (G_P(Q)/|t(Q)|)(|t(Q)|/\rho(P, Q))$. Here $G_P/|t|$ extends to a non-vanishing function on a neighborhood of P, by Section 1.2. We claim that the same is true for $|t|/\rho_P$. Namely, after making a unitary change of variables and adjusting the choice of parameter t, we may assume that $P = (1 : 0 : \cdots : 0)$, that the tangent line to X at P is given by $z_2 = \cdots = z_n = 0$, and that $t = z_1/z_0$. Thus for $Q \in X(\mathbf{C})$ with normalized coordinates $(b_0 : \cdots : b_n)$, we have $|t|\,(Q)/\rho(P, Q) = |b_1|/|b_0| \cdot |(b_1, ..., b_n)| \to 1$ as $Q \to P$. So indeed η_P extends to a global non-vanishing function, which is positive since G and ρ are.

(b) If $P \neq Q$ then $\eta(P, Q) = \eta(Q, P)$, since ρ and G are symmetric. Thus η is a symmetric, positive function on $X \times X$. Moreover, η_P is continuous for each point P, and by symmetry the function $P \mapsto \eta(P, Q)$ is continuous for each Q. Since $X(\mathbf{C})$ is compact, it follows that η is continuous and that $k^{-1} < \eta(P, Q) < k$ for some $k > 1$. ∎

Let Fun(X) be the $\mathbf{Q}$-vector space of continuous functions $f: X(\mathbf{C}) \to \mathbf{R}$, under pointwise addition, together with the sup norm. As above, regard $\mathbf{Q}$Div(X) as a $\mathbf{Q}$-vector space, under the l_1-norm.

1.3.8. LEMMA. *The homomorphism* $\mathbf{Q}\mathrm{Div}(X) \to \mathrm{Fun}(X)$ *given by $D \mapsto -\log \eta_D$ is continuous.*

Proof. In order to show that a homomorphism is continuous, it is sufficient to show that it is a bounded linear map.

If $D = \sum n_i P_i \in \mathbf{Q}\mathrm{Div}(X)$ with $|D| = d$, then for all $Q \in X(\mathbf{C})$, $\eta_D(Q) = \prod \eta(P_i, Q)^{n_i}$, and so $k^{-d} < |\eta_D(Q)| < k^d$ (with k as in the above lemma). Thus $|-\log \eta_D| < (\log k)\,|D|$, showing that the homomorphism is indeed bounded. ∎

1.3.9. THEOREM. *Let $X \subset \mathbf{P}^n$ be a smooth curve and fix a homogeneous form F_0 defining a divisor D_0. Then the multiplicative Green's pairing G on X (relative to the embedding) may be normalized so that*

(a) *If $E \in \mathbf{Q}\mathrm{CDiv}(X)$ with rE a hypersurface defined by a form F, then the constant $k(F)$ in* 1.3.2 *is equal to $|F_0|\,(D)/|F|\,(D_0)$. Thus the multiplicative Green's function $G_E = [|F_0|\,(D)/|F|\,(D_0)]\,|F|^{1/r}$.*

(b) *If* $D \in \mathbf{Q}\mathrm{Div}(X)$, *then the Green's function is given by*

$$G_D = (\lim_{\substack{E \to D \\ E \in \mathbf{Q}\mathrm{CDiv}(X)}} G_E),$$

where G_E *is given as in* (a).

Proof. This follows from the results 1.3.5–1.3.8. ∎

1.3.10. EXAMPLE. We describe Green's functions in the case that X is a plane cubic and show in particular that $G(P, Q)$ is an algebraic number provided that X, P, Q are all defined over some number field and P is a torsion point.

We consider an elliptic curve $X \subset \mathbf{P}^2$ in Weierstrass form, $y^2z = x^3 + c_2xz^2 + c_3z^3$. Thus the identity with respect to the group law is the point $O = (0:1:0)$ at infinity. Let H be the hyperplane at ∞, given by $z = 0$. Thus the divisor of H is $3 \cdot O$, and a multiplicative Green's function for H is given by $G_H(P) = |c|$, if P has normalized coordinates $(a:b:c)$. Thus a multiplicative Green's function for O is $G_O(P) = |c|^{1/3}$, for P as above. Taking the form F_0 in 1.3.9 to be z, the multiplicative Green's pairing is normalized so that $G(O, P) = G_O(P)$.

For any hyperplane H' given by a linear form $F = \alpha x + \beta y + \gamma z$, a multiplicative Green's function for F is given by $|F|\,(P) = |\alpha a + \beta b + \gamma c|$ for P as above, and so $G_{H'}(O) = \beta$. Solving for the three points P_i of $H' \cap C$ and using $G_{H'}(O) = \prod G_O(P_i)$, we may find the constant k such that $G_{H'} = kG_F$. This yields $G_{H'}$ explicitly. Note that if the coefficients of F and the coordinates of P are algebraic, then so is $|F|\,(P)$; and hence so is $G_{H'}(P)$, provided that X is defined over the algebraic numbers.

If P is a torsion point of X, the divisor P can be written as a rational sum of hyperplane sections of X, say $\sum \alpha_j H_j$. Thus the Green's function G_j of H_j is given explicitly as in the previous paragraph, and $G_P = \prod G_j^{\alpha_j}$. For example, suppose that X is the curve given in inhomogeneous coordinates by $y^2 = x^3 - x$. A Green's function for the point $O = (0:1:0)$ at ∞ is given by $(2/(|x|^2 + |x^3 - x|^2 + 1))^{1/6}$. From this, we find that a Green's function for the torsion point $P_1 = (0:0:1)$ is $(2/(|x|^2 + |x^3 - x|^2 + 1))^{1/6}\,|x|^{1/2}$, and that a Green's function for the point $P_2 = (1:0:1)$ is $|x-1|^{1/2}/2^{1/12}(|x|^2 + |x^3 - x|^2 + 1)^{1/6}$.

In general, suppose that X and P are defined over some number field. Then we may write $P = \sum \alpha_j H_j$ with each H_j defined over the algebraic numbers, and as seen above we have that $G(P, Q)$ is algebraic for all points Q on X which are defined over a number field.

Finally, given an arbitrary $Q \in C$, we may find G_Q as follows: Using the group law, find the points $Q_i = (-2)^i Q$, and let H_i be the tangent line through Q_i. Thus the divisor of H_i is $2Q_i + Q_{i+1}$, and for any $r > 0$ the divisor of $J_r := \sum_{i=0}^{r} (-1)^i 2^{-i-1} H_i$ is $Q + (-2)^{-r} Q_{r+1}$. Thus $J_r \to Q$ in the l_1 norm on the space of **Q**-divisors. Above, the multiplicative Green's function G_i for H_i was described explicitly, and we then have

$$G_Q = \prod_{i=0}^{\infty} G_i^{n_i}, \qquad \text{where} \quad n_i = (-1)^i 2^{-i-1}.$$

2. Potential Theory on Projective Arithmetic Curves

This section generalizes Section 1 to arbitrary local and global fields and considers "potential theoretic" Green's functions and Green's pairings over such fields. Subsections 2.1 and 2.2 correspond to 1.1 and 1.2, and we use the earlier results to motivate our definitions here. The heart of this section is 2.3, where we construct Green's pairings on curves, defined on arbitrary disjoint divisors, without benefit of PDEs. This also provides a new construction of Néron's pairing on curves. While we emphasize Green's pairings arising from projective embeddings, we also prove a more general existence result. Section 2.4 then shows that there is an induced global pairing which satisfies properties similar to those of intersection pairings.

2.1. *Projective Geometry over Local Fields*

Let K be a local field, viz. a finite extension of a base field F which is equal to $\mathbf{R}$, $\mathbf{Q}_P$, or $\mathbf{F}_p((t))$. Give the base field $\mathbf{F}$ the usual absolute value $|\ |_F$. That is, $|\ |_F$ is the Euclidean absolute value if K is archimedean, and otherwise it is the absolute value on $\mathbf{Q}_p$ (resp. on $\mathbf{F}_p((t))$) which sets $|p|$ (resp. $|t|$) equal to p^{-1}. On K there is a unique absolute value $|\ |_K$ extending $|\ |_F$, and K is complete and locally compact with respect to $|\ |_K$. Also, for $a \in K$ we set $\|a\|_K = |a|^{[K:F]} = |\mathbf{N}_{K/F} a|_F$, where $\mathbf{N}_{K/F}$ is the norm map from K to F. (Thus any *global* field satisfies the usual product formula with respect to these norms $\|\ \|_v$ on each completion K_v.) Then $\|\ \| = \|\ \|_K$ defines a *norm* on K, i.e., a non-negative real-valued function on K such that $\|a\| > 0$ for $a \neq 0$, $\|ab\| = \|a\|\ \|b\|$, and $\|a+b\| \leqslant 2\max(\|a\|, \|b\|)$.

We will define a distance function on $\mathbf{P}^n(K_a)$, where K_a is the algebraic closure of K. In the case $K = \mathbf{R}$, this will agree with the chordal distance on complex projective space defined in Section 1.1. In the non-archimedean case, this distance function will have an interpretation in terms of the geometry of $\mathbf{P}^n(O)$, where O is the valuation ring of K (viz. O is the integral closure of $\mathbf{Z}_p$ or $\mathbf{F}_p[[t]]$ in K).

First, following Vojta [Vo, Definition 4.12, Ex. 4.14, p. 66], we define a norm on affine n-space. Namely, for any finite extension L of K, and

$\mathbf{a}=(a_1, ..., a_n) \in L^n$, define $\|\mathbf{a}\|_{1,L}=\|a_1\|_L+\cdots+\|a_n\|_L$ and let $\|\mathbf{a}\|_{L/K}=\|\mathbf{a}\|_{1,L}^{1/[L:K]}$. Then for any $\mathbf{a}=(a_1, ..., a_n) \in K_a^n$, consider the direct system of those L, finite over K, such that $\mathbf{a} \in L^n$. Let $\|\mathbf{a}\|_K=\lim_{L \to K_a} \|\mathbf{a}\|_{L/K}$; note that this exists and is equal to the l_q-norm on K^n (relative to the norm $\| \; \|_K$ on K), where $q=[K_a : K]$. Finally, for $\mathbf{a} \in F_a^n=K_a^n$, let $|\mathbf{a}|=\|\mathbf{a}\|_F$, so that $|\mathbf{a}|=\|\mathbf{a}\|^{1/[K:F]}$. (Note that in the case $n=1$, this agrees with the previous notation for the absolute value $| \; |$ on K.)

2.1.1. Example. (a) If $K=\mathbf{R}$, then $\| \; \|_K=| \; |$ is just the usual Euclidean norm on $\mathbf{R}^n$, while if $K=\mathbf{C}$, then $| \; |_K$ is the Euclidean norm and $\| \; \|_K=| \; |^2$ is the square of the Euclidean norm.

(b) Say K is non-archimedean, with valuation ring O having maximal ideal $\mathbf{m}$. Then $\|\mathbf{a}\|_K=\max\{\|a_i\|_K : i=1, ..., n\}$ and $|a|=\max\{|a_i|_K : i=1, ..., n\}$, where $\| \; \|_K$ and $| \; |_K$ are the two normalizations of the $\mathbf{m}$-adic absolute value. Thus two points are close precisely when they are congruent modulo a high power of $\mathbf{m}$.

2.1.2. Proposition. *Let K be a local field, and give K^n the above norm $| \; |_K$. Then $\{A \in GL_n(K) \mid A$ induces an isometry of $K^n\}$ is a maximal compact subgroup $\Gamma_n(K) \subset GL_n(K)$. Namely, $\Gamma_n(K)$ is the orthogonal group $O(n)$ if $K=\mathbf{R}$; the unitary group $U(n)$ if $K=\mathbf{C}$; and the group $GL_n(O)$ if K is non-archimedean with valuation ring O.*

Proof. If $K=\mathbf{R}$ or $\mathbf{C}$, this follows immediately from Example 2.1.1(a). In the non-archimedean case, $| \; |$ is the sup norm on the coefficients. Since O^n is the closed unit ball in K^n, it easily follows that the isometry group is $GL_n(O)$. In all cases, the isometry group is indeed maximally compact in $GL_n(K)$. ∎

Note that we may identify $\Gamma_1(K)=\{a \in K \mid |a|=1\}$. For example, this is the circle group if $K=\mathbf{C}$. Note also that $\Gamma_1(K)$ acts on $\Gamma_n(K)$ by scalar multiplication. We denote the quotient $\Gamma_n(K)/\Gamma_1(K)$ by $P\Gamma_n(K)$.

Given a point $P \in \mathbf{P}^n(K_a)$, say that a choice of homogeneous coordinates $(a_0 : \cdots : a_n)$ for P is *normalized* if $|(a_0, ..., a_n)|=1$ in K_a^{n+1}. Motivated by Section 1.1, we consider the metric (i.e., distance function) $\rho=\rho_K$ on $\mathbf{P}^n(K_a)$ given by

$$\rho(P, Q)=|(..., a_i b_j-a_j b_i, ...)_{i \leqslant j}|,$$

where P, Q have normalized coordinates $(a_0 : \cdots : a_n)$, $(b_0 : \cdots : b_n)$, respectively, and where the norm $| \; |=| \; |_K$ is on K_a^N, $N=n(n+1)/2$. Note that the metric ρ on $\mathbf{P}^n(K_a)$ has diameter 1.

2.1.3. EXAMPLE. (a) If $K=\mathbf{R}$ or $\mathbf{C}$, so that $K_a=\mathbf{C}$, then ρ_K is equal to the chordal distance on $\mathbf{P}^n(\mathbf{C})$.

(b) Suppose K is non-archimedean, with valuation ring O, having uniformizer π. Given $P, Q \in \mathbf{P}^n(K)$, let the 1-cycles $c(P)$, $c(Q)$ in $\mathbf{P}^n_O$ be their closures. These are curves, and let $i(P, Q)$ is the order of contact of $c(P)$ with $c(Q)$ in $\mathbf{P}^n_O$. Thus $i(P, Q) \leqslant r$ if and only if there are elements $a_j, b_j \in O$ with $P=(a_0 : \cdots : a_n)$, $Q=(b_0 : \cdots : b_n)$, and $a_j \equiv b_j \pmod{\pi^r}$. Then $i(P, Q) = -\log_q \rho(P, Q)$, where $q=|\pi|^{-1}>1$.

As in the special case of $K=\mathbf{C}$, for any local field K there is a *Hopf bundle* $\pi\colon H_n \to \mathbf{P}^n(K)$, where $H_n=\{A \in K^{n+1} \mid |A|=1\}$. This is a principal bundle for the group $\Gamma_1(K)$, and making a choice of normalized coordinates for $P \in \mathbf{P}^n(K)$ is equivalent to choosing a point A on H_n over P.

2.1.4. PROPOSITION. *Let K be a local field, and give $\mathbf{P}^n(K)$ the above distance function ρ. Then $\{X \in PGL_{n+1}(K) \mid X$ induces an isometry of $\mathbf{P}^n(K)\}$ is a maximal compact subgroup of $PGL_{n+1}(K)$, viz. $P\Gamma_{n+1}(K) = \Gamma_{n+1}(K)/\Gamma_1(K)$. Thus this isometry group is $PO(n+1)=O(n+1)/\pm 1$ if $K=\mathbf{R}$; $PU(n+1)=U(n+1)/S^1$ if $K=\mathbf{C}$; and $PGL_{n+1}(O)$ if K is non-archimedean with valuation ring O.*

Proof. The real and complex cases follow from the interpretation in terms of the chordal distance function of Section 1. In the non-archimedean case, every element of $PGL_{n+1}(O)$ induces an automorphism over O of $\mathbf{P}^n_O$, and hence by Example 2.1.3(b) it lies in the isometry group. Conversely, suppose $X \in PGL_{n+1}(K)$ induces an isometry. By multiplying by an element of $PGL_{n+1}(O)$, we may assume that X fixes the point $(1:0:\cdots:0)$. Since X gives an isometry, it leaves invariant $\{P \in \mathbf{P}^n(K) \mid \rho(P, Q)=1\}$, i.e., the hyperplane $H: z_0=0$. Identifying H with $\mathbf{P}^{n-1}(K)$, the result follows by induction. ∎

We then have the following three results, which are generalizations of 1.1.5–1.1.7 to arbitrary local fields K. The proofs are the same as for 1.1.5–1.1.7.

2.1.5. PROPOSITION. (a) *Let $P, Q \in \mathbf{P}^n(K)$, let $A \in H_n$ be a unit vector of K^{n+1} lying over P, and let L_Q be the K-line in K^{n+1} corresponding to Q. Then $\rho(P, Q)$ is equal to the distance from A to L_Q in K^{n+1}.*

(b) *Thus $\mathbf{P}^n$ has diameter 1, relative to ρ.*

2.1.6. COROLLARY. *Let $X \subset K^{n+1}$ be a vector subspace and let $A \in H_n$ be a unit vector in K^{n+1}, and let $H \subset \mathbf{P}^n(K)$ and $P \in \mathbf{P}^n(K)$ be their images in* *projective space. Then the distance from P to H in $\mathbf{P}^n(K)$ (i.e., $\min\{\rho(P, Q) \mid Q \in H\}$) is equal to the distance from A to X in K^{n+1}.*

Following Section 1.1, if F is a homogeneous polynomial in $z_0, ..., z_n$, and $A \in H_n \subset K^{n+1}$ is a unit vector lying above $P \in \mathbf{P}^n(K)$, we define $|F|\,(P) = |F(A)|$. Given P, this is independent of the choice of A. Also, if F has degree d and $P = (a_0 : \cdots : a_n)$, then $|F|\,(P) = |F(a_0, ..., a_n)|/|(a_0, ..., a_n)|^d$.

2.1.7. PROPOSITION. *Let H be a hyperplane in $\mathbf{P}^n(K)$, defined by the vanishing of a linear form $F = \sum_{j=0}^n \alpha_j z_j$, normalized so that $|(\alpha_0, ..., \alpha_n)| = 1$. Then for $P \in \mathbf{P}^n(K)$, the distance from P to H is equal to $|F|\,(P)$.*

2.2. *Green's Functions over Local Fields*

Motivated by the results of Section 1.2, we consider Green's functions associated to divisors in projective space over local fields K. Specifically, let F be a homogeneous form over K in variables $z_0, ..., z_n$, and let H be the hypersurface in $\mathbf{P}^n(K)$ defined by F. Thus we have $|F|: \mathbf{P}^n(K_a) \to \mathbf{R}$, a non-negative valued continuous function which vanishes precisely on H and which is locally bounded (i.e., is bounded when restricted to subsets of $\mathbf{P}^n(K_a)$ which are ρ-bounded or, equivalently, which are $|\ |$-bounded in terms of the inhomogeneous coordinates in an affine patch). Following Proposition 1.2.4, we define the function $|F|$ to be the *multiplicative Green's function* associated to F and say that $|F|$ is a *multiplicative Green's function* associated to the hypersurface H. We say that $g = -\log G: \mathbf{P}^n(K) \to \mathbf{R}$ is the *Green's function* associated to the form F (resp. to the hypersurface H), and write $g = g_{|F|}$. Also, if F is a form defining H as above, then for any $c \in \mathbf{R}$ we will call the function $g_{|F|} + c$ a *Green's function* associated to the hypersurface H. (This definition is independent of the choice of F, since F is determined up to multiplication by a non-zero constant.) For any form F the function $a \mapsto g_{|F|}(a)$ is a locally bounded continuous function on $X - \operatorname{Supp}(D)$, where D is the divisor of F. Also, for any $a \in X(K_a)$ the function $F \mapsto g_{|F|}(a)$ is continuous as a function on $\{$homogeneous forms of degree d in $X_0, ..., X_n\} \approx K^N$, where $N = \binom{d+n}{n}$ and K^N is given the metric $|\ |$.

Note that if E and F are homogeneous forms, then $|EF| = |E|\,|F|$ is an equality of multiplicative Green's functions. If $F = c \in K^*$ is of degree 0, then the multiplicative Green's function $|c|$ is the constant function with value equal to the norm of the element c. If $c \in K^*$ and $E = cF$, then $|E| = |c|\,|F|$, so $|E| = |F|$ if and only if $|c| = 1$.

Extending by linearity, we may define *Green's functions* g_D for all (not necessarily effective) divisors D on $\mathbf{P}^n_K$, each of which is determined uniquely up to adding a constant. Also, by linearity we may evaluate Green's functions g_D on zero cycles $a \in Z(\mathbf{P}^n(K))$, where the supports of $D=(F)$ and a are required to be disjoint. Thus we obtain a bilinear pairing $(D, a) \mapsto g_D(a)$.

2.2.1. PROPOSITION. *Let D be a hypersurface in $\mathbf{P}^n_{\mathbf{C}}$, and g a Green's function for D (in the above sense). Then g is a Green's function for D with respect to the Fubini–Study metric on $\mathbf{P}^n(\mathbf{C})$ (in the sense of Section 1.2).*

Proof. By Proposition 1.2.4. ∎

Also, this generalization of 1.2.5 holds:

2.2.2. PROPOSITION. *If H is a hyperplane in $\mathbf{P}^n(K)$, the function*

$$G(P) = \textit{distance from } P \textit{ to } H$$

is a multiplicative Green's function on $\mathbf{P}^n(K)$ for H.

Proof. By the definition of multiplicative Green's functions together with 2.1.7. ∎

More generally, suppose that X is a projective variety over a local field K, and let D be a Cartier divisor on X. If $\Phi\colon X(K_a) \to \mathbf{R}$ is a continuous non-negative function which vanishes precisely at supp Y, we say that Φ is a *distance function* for D in X if for every Zariski open set $U \subset X$ and every $h \in K(U)^*$ which is a local equation for D, the function Φ/h extends to a continuous locally bounded non-vanishing real-valued function on $U(K_a)$. Also, call $\phi = -\log \Phi\colon (X - \operatorname{supp} D)(K_a) \to \mathbf{R}$ a *Weil function* (or *logarithmic distance function*) for D in X [La, Chap. 10]. That is, a function $\phi\colon (X - \operatorname{supp} D)(K_a) \to \mathbf{R}$ is a Weil function for D if for (U, h) as above, the function $\phi + \log |h|$ extends to a continuous locally bounded real-valued function on $U(K_a)$. A Weil function for D can also be regarded as a linear function on the zero cycles of $X(K_a)$ whose support is disjoint from that of D, since each ϕ as above corresponds to such a function. The Weil functions on X form a group $\operatorname{Weil}(X)$, and there is a canonical homomorphism $\operatorname{Weil}(X) \to \operatorname{Div}(X)$, assigning each Weil function to its associated Cartier divisor. Let $\operatorname{Weil}^*(X)$ be the quotient by the constant functions (which are Weil functions for the trivial divisor). Thus there is the induced map $\pi\colon \operatorname{Weil}^*(X) \to \operatorname{Div}(X)$. Writing $\mathbf{Q}\operatorname{Div}(X) = \operatorname{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, $\mathbf{Q}\operatorname{Weil}(X) = \operatorname{Weil}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, and $\mathbf{Q}\operatorname{Weil}^*(X) = \mathbf{Q}\operatorname{Weil}(X)/(\text{constants})$, we also obtain a homomorphism $\pi\colon \mathbf{Q}\operatorname{Weil}^*(X) \to \mathbf{Q}\operatorname{Div}(X)$, and we say that an element $\phi \in \mathbf{Q}\operatorname{Weil}(X)$ is a *Weil function* for the induced $\mathbf{Q}$-divisor in $\mathbf{Q}\operatorname{Div}(X)$.

Also, if Div′ is any subgroup of $\mathbf{Q}\,\mathrm{Div}(X)=\mathrm{Div}(X)\otimes_{\mathbf{Z}}\mathbf{Q}$, we say that a *Weil form* on Div′ is a homomorphism $\phi\colon \mathrm{Div}'\to \mathbf{Q}\mathrm{Weil}^*(X)$ which is a section of π over Div′, and such that if $D=(f)$ is a principal divisor then $\phi(D)=-\log|f|$ (modulo constant functions). Note that if $\phi\colon D\mapsto\phi_D$ is a Weil form on Div′ then each ϕ_D is determined exactly (and not just up to a constant) on zero cycles of degree zero. And indeed, giving a Weil form on Div′ is equivalent to giving a real-valued bilinear function on $\mathrm{Div}'\times Z_0X(K_a)-\Delta$ (where $Z_0X(K_a)$ consists of zero cycles of degree zero on $X(K_a)$ and Δ is the set of pairs with non-disjoint support) such that

(i) for each $D\in\mathrm{Div}'$ and each $P\in X(K_a)-(\mathrm{supp}\,D)$ the function $z\mapsto\langle D, z-(\deg z)P\rangle$ is a Weil function for D, and

(ii) if $f\in K(X)$ then $\langle (f), P\rangle = -\log|f|\,(P)$ (up to a constant).

2.2.3. Proposition. *Let K be a local field, and for every divisor D on $\mathbf{P}^n_K$ let g_D be a Green's function for D on $\mathbf{P}^n_K$. Then the association $D\mapsto g_D$ is a Weil form on $\mathbf{P}^n_K$ which is invariant under the action of the isometry group $P\Gamma_{n+1}(K)$ of $\mathbf{P}^n_K$.*

Proof. As noted above, to show that $D\mapsto g_D$ is a Weil form it suffices to show that the induced pairing on $\mathrm{Div}(\boldsymbol{P}^n_K)\times Z_0\boldsymbol{P}^n_K(K_a)-\Delta$ satisfies (i) and (ii) above. Property (ii) is immediate from the definition of g_D. For (i), let $f\in K(\boldsymbol{P}^n_K)$ be a local equation for a divisor D in a Zariski open set U, and g_D be a Green's function for D. Thus $g_D=-\log|F|$, where F is a homogeneous form defining D. So $g_D+\log|f|$ is continuous and locally bounded on $U(K_a)$. Thus this is a Weil form. Finally, since a $\Gamma_{n+1}(K)$-change of variables takes normalized coordinates to normalized coordinates (by Proposition 2.1.2), an element of $P\Gamma_{n+1}(K)$ takes Green's functions to Green's functions. ∎

As in Section 1.2, if $i\colon N\hookrightarrow \boldsymbol{P}^n_K$ is an embedding and D is a divisor on N, call D a *complete intersection divisor* on N (relative to i) if $D=i^*(H)$ for some hypersurface H in $\mathbf{P}^n$, i.e., if and only if D is the zero locus on N of a homogeneous form F. For a fixed embedding i, these divisors form a subgroup $\mathrm{CDiv}_i(N)\subset\mathrm{Div}(N)$. (If the choice of embedding i is understood, it will be suppressed in the notation.) Also, define the group of complete intersection $\mathbf{Q}$-divisors $\mathbf{Q}\mathrm{CDiv}_i(N)\subset\mathbf{Q}\mathrm{Div}(N)$ to be $\mathrm{CDiv}_i(N)\otimes_{\mathbf{Z}}\mathbf{Q}$. Thus a $\mathbf{Q}$-divisor D is an element of $\mathbf{Q}\mathrm{CDiv}_i(N)$ if and only if some non-zero integral multiple rD is linearly equivalent to a multiple of i^*H, where H is a generic hyperplane section of the ambient $\mathbf{P}^n$. Note that an element $D\in\mathbf{Q}\mathrm{CDiv}_i(N)$ is *numerically equivalent* to zero (i.e., some non-zero multiple of D is an actual divisor which is numerically equivalent to zero) if and only if some non-zero multiple of D is linearly equivalent to zero.

Following Lemma 1.2.2, if $i: N \hookrightarrow \mathbf{P}^n$ and H is a hypersurface in $\mathbf{P}^n$, we define a *Green's function* on N for the complete intersection hypersurface $D = H \cap N$ (relative to this embedding) to be the restriction of a Green's function on H in $\mathbf{P}^n$. Extending by linearity yields a notion of Green's functions on $\mathbf{Q}\mathrm{CDiv}_i(N)$. Note that these Green's functions are unique up to adding a constant, since the same is true for divisors in $\mathbf{P}^n$. Define a *multiplicative Green's function* associated to D to be a function of the form $G = \exp(-g)$, where g is a Green's function for D. The analog of 1.2.6 then holds: the function $G(P) = |F|\,(P)$ is a multiplicative Green's function on $N \subset \mathbf{P}^n$ for the zero locus of a homogeneous form F. Also, as for $\mathbf{P}^n$, we may regard G as a function on L-valued points of N, where L is a finite extension of K.

Caution. As in Section 1, the Green's function for a divisor D on a variety $N \subset \mathbf{P}^n$ depends not only on the *isomorphism* class of the embedding of N into projective space, but also on the *isometry* class. Thus if two embeddings differ by an element $A \in PGL_{n+1}(K)$, then they induce the same set of complete intersection divisors on N; but they will also induce the same Green's functions if and only if $A \in P\Gamma_{n+1}(K)$. Cf. also [Ha, Proposition 1.12], for a discussion of the non-archimedean case.

2.2.4. PROPOSITION. *Let K be a local field, let $i: W \hookrightarrow V$ be an embedding of K-schemes, let $\mathrm{Div}' \subset \mathrm{Div}(V)$ be a subgroup, and let $\phi: D \mapsto \phi_D$ be a Weil form on $\mathbf{Q}\mathrm{Div}' \subset \mathbf{Q}\mathrm{Div}(V)$. Suppose that $i^*: (\mathrm{Div}')/\sim \;\to \mathrm{Pic}\, W$ is injective, where $\sim$ denotes linear equivalence. Set $\mathbf{Q}\mathrm{Div}'' = i^*(\mathbf{Q}\mathrm{Div}') \subset \mathbf{Q}\mathrm{Div}(W)$, and for $D \in \mathbf{Q}\mathrm{Div}'$ define $(i^*\phi)_{D''}: P \mapsto \phi_D(i(P))$ modulo constants, if $i^*D = D''$. Then $i^*\phi$ is a Weil form on $\mathbf{Q}\mathrm{Div}''$.*

Proof. The only difficulty is whether $D'' \mapsto (i^*\phi)_{D''}$ is well defined. So suppose that $i^*D = i^*D' = D''$. We may assume $D, D' \in \mathrm{Div}'$. By the injectivity hypothesis, $D \sim D'$. So there is a rational function $f \in K(V)^*$ such that $D' = D + \mathrm{div}(f)$, and thus $\phi_{D'} = \phi_D - \log|f|$ modulo constants. Also, $\mathrm{div}(i^*(f)) = i^*D' - i^*D = 0$, and so the rational function $i^*(f)$ is constant. Thus $\phi_D \circ i = \phi_{D'} \circ i$ modulo constants, and so $i^*\phi$ is a well-defined Weil form. ∎

Remark. Cf. also Manin's counterexample [Ma, Section 2] to [Ne, Chap. II, Theorem 2], concerning the necessity of injectivity on Pic.

We have the following analog of 1.2.3:

2.2.5. COROLLARY. *Given any projective non-singular K-variety N and embedding $i: N \hookrightarrow \mathbf{P}^n_K$, for each divisor $D \in \mathbf{Q}\mathrm{CDiv}_i(N)$ let $g_D = g_{D,N,i}$ be an associated Green's function (which is determined up to a constant). Then*

(a) *The map* $D \mapsto g_D$ *is a Weil form on* $\mathbf{Q}\mathrm{CDiv}_i(N)$.

(b) *If* $D \in \mathbf{Q}\mathrm{CDiv}_i(N)$ *is numerically equivalent to zero, then* g_D *depends only on D and N, and not on the embedding i.*

(c) *Let* $\mu: N \to M$ *be a morphism of projective varieties. Suppose* $D \in \mathbf{Q}\mathrm{Div}(M)$ *is algebraically equivalent to zero and is a complete intersection* $\mathbf{Q}$*-divisor with respect to some projective embedding. Suppose also that* $C = \mu^{-1}(D)$ *is a divisor on N. Then* $g_{C,N} = g_{D,M} \circ \mu$ *modulo constant functions (where the Green's functions are with respect to any choices of projective embedding).*

Proof. Assertion (a) follows from Proposition 2.2.4, using that divisors of degree 0 in $\mathbf{P}^n$ are all linearly equivalent to 0. Assertion (b) follows from (a), since D is necessarily linearly equivalent to zero (as observed prior to the statement of Proposition 2.2.4). In (c), let $i: M \subsetneq \mathbf{P}^n$ be the asserted embedding and let $j = i \circ \mu$. Then $g_{C,N,i} = g_{D,M} \circ \mu$ modulo a constant function, by definition; and by (b), the same is true for Green's functions with respect to any embedding. ∎

2.2.6. Proposition. *Let N be a smooth projective variety over* $\mathbf{C}$, *let* $i: N \subsetneq \mathbf{P}^n_{\mathbf{C}}$ *be an embedding, and let* $D \in \mathrm{Div}\, N$ *be a complete intersection* $\mathbf{Q}$*-divisor on N. Then a Green's function for D on N with respect to the embedding (in the above sense) is a Green's function for D on* $N(\mathbf{C})$ *with respect to the pullback to* $N(\mathbf{C})$ *of the Fubini–Study metric on* $\mathbf{P}^n(\mathbf{C})$ *(in the sense of Section* 1.2*).*

Proof. By 2.2.1 and 1.2.2. ∎

Remarks. (a) Corollary 2.2.5, unlike Proposition 1.2.3, restricts attention to complete intersection divisors on varieties $N \subset \mathbf{P}^n_K$. This is due to the fact that we have defined Green's functions only for these divisors. It would be preferable to extend the definition and proposition to more general divisors, which in particular would give a new construction of Néron functions on arbitrary divisors algebraically equivalent to zero (cf. the comments after 1.2.3). But in the case that N is a curve, this extension will be accomplished in Section 2.3, thus giving in particular a construction of the Néron height pairing.

(b) As in the complex case, multiplicative Green's functions for hypersurfaces in $\mathbf{P}^n_K$ actually arise from K-valued functions on the Hopf bundle. Namely, if H is the hypersurface defined by a form F, then G_H is the (descent to $\mathbf{P}^n$ of) the absolute value of the function F on the total space H_n of the Hopf bundle $\pi: H_n \to \mathbf{P}^n(K)$ for K (cf. Section 2.1). And if D is a complete intersection hypersurface in $N \subset \mathbf{P}^n_K$, say with $D = H \cap N$ and $H = (F)$ in $\mathbf{P}^n$, then the associated multiplicative Green's function

$G = G_{D,N}$ is induced by the function F on the total space $\pi^{-1}(N)$ of the Hopf bundle $\pi^{-1}(N) \to N$ over N. ▮

2.3. *Green's Pairings on Curves over Local Fields*

The main result of this section is that Weil forms on curves over local fields K induce bilinear symmetric pairings on divisors (of *arbitrary* degree) which in degree 0 agree with the Néron height pairing. In particular, taking as the Weil form the one considered in Section 2.2 (obtained by Green's functions on an ambient projective space), we obtain a pairing which in the case of $K = \mathbf{C}$ agrees with Arakelov's pairing with respect to the Fubini–Study metric. Alternatively, taking instead a Weil form induced by the Néron pairing on the Jacobian, we obtain a pairing which in the special case $K = \mathbf{C}$ agrees (at least for a generic curve) with the Arakelov pairing relative to the canonical Hermitian metric on the curve. Thus we obtain new constructions for each of these Arakelov pairings in the case of $K = \mathbf{C}$, and obtain generalizations of these pairings to the case of arbitrary local fields. These new pairings extend Néron's pairing on degree zero divisors on the curve.

We will give two proofs of this result. The first proof has the advantage of not relying on any prior information about the Néron pairing on curves, and indeed it provides a new construction of that pairing. The second proof relies on the existence of Néron's pairing, but it has the advantage of being shorter. Both proofs begin with the following "triple reciprocity" lemma:

2.3.1. LEMMA. *Let X be a smooth projective curve over a local field K, let $i\colon X \subsetneq \mathbf{P}^n_K$ be an embedding, and let $\phi\colon \mathrm{QCDiv}_i(X) \to \mathrm{Weil}^*(X)$ be a Weil form. Let D_0, D_1, $D_2 \in \mathrm{QCDiv}_i(X)$ be linearly equivalent divisors on X having disjoint support* (*where the indices are modulo* 3). *For each j let ϕ_j be a lifting of $\phi(D_j)$ to a Weil function for D_j. Then*

$$\sum_{j=0}^{2} \phi_j(D_{j+1}) = \sum_{j=0}^{2} \phi_j(D_{j-1}).$$

Proof. Since the divisors D_j are linearly equivalent, and since ϕ is linear, we are reduced to the case that D_0, D_1, D_2 are respectively defined by homogeneous polynomials F_0, F_1, F_2 in $z_0, ..., z_n$ of common degree $d > 0$. Let $f_j = i^*(F_{j+1}/F_{j-1}) \in K(X)$. Thus the divisor of f_j is $D_{j+1} - D_{j-1}$, and $\prod f_j = 1$. Since ϕ is a Weil form, it follows that for each j there is a constant $c_j \in \mathbf{R}$ such that $\phi_{j+1} - \phi_{j-1} = -\log |f_j| + c_j$. Note that $\sum c_j = 0$. Thus it suffices to show that $\prod |f_j|\,(D_j) = 1$, i.e., that $\prod |F_j|\,(D_{j+1}) = \prod |F_j|\,(D_{j-1})$.

Consider the morphism $\pi\colon (F_0 : F_1 : F_2)\colon X \to \mathbf{P}^2$, given by $P \mapsto (F_0(P) : F_1(P) : F_2(P))$. This is well defined, since the supports of the D_j are disjoint, and it is non-constant, since $d > 0$. Let $m = \deg \pi$. The image $\pi(X)$ is defined by a homogeneous polynomial $R(z_0, z_1, z_2)$, say of degree s. Here $|F_0|\ (D_1) \neq 0$ since D_0 and D_1 have disjoint support, and if $D_1 = \subset n_k P_k$, then

$$\begin{aligned} |F_2|\ (D_1)/|F_0|\ (D_1) &= \prod_k |z_2/z_0|\ (\pi(P_k))^{n_k} \\ &= |[\text{product of the roots of } R(1, 0, z)]|^m \\ &= |C_0/C_2|^m, \end{aligned}$$

where C_i is the coefficient of z_i^s in R. Similarly, $|F_1|\ (D_0)/|F_2|\ (D_0) = |C_2/C_1|^m$ and $|F_0|\ (D_2)/|F_1|\ (D_2) = |C_1/C_0|^m$. So $\prod |F_j|\ (D_{j+1}) = \prod |F_j|\ (D_{j-1})$, as desired. ∎

Remark. Lemma 2.3.1 is equivalent to the assertion that if F_0, F_1, F_2 are homogeneous forms of the same degree on $\mathbf{P}_K^n$ which define divisors D_0, D_1, D_2 on a curve $X \subset \mathbf{P}_K^n$, then $\prod |F_i|\ (D_{i+1}) = \prod |F_i|\ (D_{i-1})$. The analogous assertion, with four rather than three divisors, is equivalent to Weil reciprocity. ∎

Next, we prove a result which, in the case that the Weil form is given by Green's functions relative to the embedding, corresponds in the complex analytic case to Lemma 1.3.6(a) (although here we restrict attention to complete intersection divisors).

2.3.2. Lemma. *Let X be a smooth projective curve in $\mathbf{P}_K^n$.*

(a) *If $D \in \mathbf{Q}\mathrm{Div}(X)$, define $\rho_D\colon X(K_a) \to \mathbf{R}$ by $\rho_D(Q) = \prod \rho(P_j^{b_j}, Q)$, where $D = \sum a_j P_j$ with each $P_j \in X(K_a)$. Then ρ_D is a distance function on X (cf. Section 2.2).*

(b) *Let ϕ_D be a Weil function on a $\mathbf{Q}$-divisor $D \in \mathbf{Q}\mathrm{Div}(X)$ and let $\Phi_D = \exp(-\phi_D)$. Then the function $\eta_D = \Phi_D/\rho_D\colon X(K_a) - (\mathrm{supp}\, D) \to \mathbf{R}$ extends to a continuous positive valued function $X(K_a) \to \mathbf{R}$ whose logarithm is bounded.*

Proof. Part (b) follows from (a), because the difference of Weil functions is a Weil function. For part (a), we may replace K by a finite extension over which each of the points P_j is defined, and then by linearity of $-\log \rho_D$ we are reduced to the case that $D = P$, a single K-valued point with multiplicity 1. We now proceed as in the proof of 1.3.6(a). Namely, let t be a local parameter on X at P. After making an isometric change of variables (i.e., adjusting the embedding $X \hookrightarrow \mathbf{P}_K^n$ by an element of $P\Gamma_{n+1}(K)$, which does not affect the Green's function) and adjusting the choice of parameter t, we may assume that $P = (1 : 0 \cdots : 0)$, that the tangent line to

X at P is given by $z_2 = \cdots = z_n = 0$, and that $t = z_1/z_0$. Thus for $Q \in X(K_a)$ with normalized coordinates $(b_0 : \cdots : b_n)$, we have $|t|\,(Q)/\rho(P, Q) = |b|/|b_0|\,\|(b_1, ..., b_n)|$, and this approaches 1 as $Q \to P$. So indeed $|t|/\rho_P$ extends to a global non-vanishing function which is continuous and bounded and which is positive, since ϕ and ρ are. ▌

Now suppose that $X \subset \mathbf{P}^n_K$ is a smooth projective curve and that ϕ: $\mathrm{Div}' \to \mathbf{Q}\mathrm{Weil}^*(X)$ is a Weil form on a subgroup $\mathrm{Div}' \subset \mathbf{Q}\,\mathrm{Div}(X)$. For any lifting $D \mapsto \phi_D$ of ϕ to a map $\mathrm{Div}' \to \mathbf{Q}\mathrm{Weil}(X)$, Lemma 2.3.2 assigns to each $D \in \mathrm{Div}'$ a continuous bounded positive function η_D: $X(K_a) \to \mathbf{R}$. Extending $-\log \eta_D$ by linearity to the group $\mathbf{Q}ZX(K_a) = ZX(K_a) \otimes_{\mathbf{Z}} \mathbf{Q}$ of $\mathbf{Q}$-zero cycles on $X(K_a)$, we obtain for each $D \in \mathrm{Div}'$ a map η_D: $\mathbf{Q}ZX(K_a) \to \mathbf{R}$ such that $-\log \eta_D$ is a homomorphism. This yields a positive valued map η: $\mathrm{Div}' \times \mathbf{Q}ZX(K_a) \to \mathbf{R}$ given by $\eta(D, Z) = \eta_D(Z)$, such that $-\log \eta$ is bilinear.

2.3.3. Lemma. *Let X be a smooth projective curve over a local field K, let i: $X \subsetneq \mathbf{P}^n_K$ be an embedding, and let ϕ:* $\mathbf{Q}\mathrm{CDiv}_i(X) \to \mathrm{Weil}^*(X)$ *be a Weil form. Then there exists a homomorphism* $\mathbf{Q}\mathrm{CDiv}_i(X) \to \mathrm{Weil}(X)$ *lifting ϕ such that the induced map η:* $\mathbf{Q}\mathrm{CDiv}_i(X) \times \mathbf{Q}ZX(K_a) \to \mathbf{R}$ *is a positive function, such that $-\log \eta$ is bilinear and such that the restriction to* $\mathbf{Q}\mathrm{CDiv}_i(X) \times \mathbf{Q}\mathrm{CDiv}_i(X)$ *is symmetric. Moreover, η is unique up to multiplication by a non-zero constant.*

Proof. Let H be the pullback to X of a hyperplane in $\mathbf{P}^n_K$, and choose a corresponding Weil function ϕ_H. Thus, as in 2.3.2, we obtain an induced function η_H on $X(K_a)$, which as above we extend to a function on zero cycles over K_a. For each $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$ the Weil function for D is determined only up to a constant; choose ϕ_D so that the induced function η_D satisfies $\eta_D(H) = \eta_H(D)$. Then the function $\eta(D, D') = \eta_D(D')$ satisfies $\eta(D, H) = \eta(H, D)$, for all $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$.

Thus for all $D \in \mathbf{Q}\mathrm{CDiv}(X)$, the function $Z \mapsto \eta(D, Z)$ is a function η_D as in 2.3.2. In order to show symmetry, it suffices to check the case of pairs (D_0, D_1) in which D_0, $D_1 \in \mathrm{CDiv}_i(X)$ are effective. Since $\eta(D, H) = \eta(H, D)$, after adding a multiple of H to D_0 or D_1, we may assume that D_0 and D_1 are of the same degree d. Say $D_1 = (G_1)$, $D_2 = (G_2)$. Also write $D_0 = dH$, $G_0 = F^d$. By continuity of each η_D as a function on points (cf. 2.3.2), we may assume that D_0, D_1, D_2 have pairwise disjoint support. Symmetry now follows from Lemma 2.3.1, and the definition of η_D.

Finally, uniqueness follows from the uniqueness of η_D up to multiplication by a constant, together with symmetry. ▌

We will use the symmetry of η on complete intersection divisors to extend η to pairs of more general divisors, having disjoint support.

As in Section 1, given a curve X we may put an l_1-norm on $\mathbf{Q}\mathrm{Div}(X)$, by setting $|\sum b_j x_j|_1 = \sum |b_j d_j|$, where each x_j is a closed point of X whose residue field has degree d_j over K. Equivalently, $|\sum a_j P_j|_1 = \sum |a_j|$, where each $P_j \in X(K_a)$. Then the group $\mathbf{Q}\mathrm{CDiv}_i(X)$ of complete intersection $\mathbf{Q}$-divisors on X is dense in the full group $\mathbf{Q}\mathrm{Div}(X)$; indeed the proof of Lemma 1.3.5 carries over to more general local fields. Implicitly making use of this density, we obtain the following generalization of Lemma 1.3.6(b):

2.3.4. PROPOSITION. *Let X be a smooth complete curve over a local field K, let $i\colon X \subsetneq \mathbf{P}^n_K$ be an embedding, and let $\phi\colon \mathbf{Q}\mathrm{CDiv}_i(X) \to \mathrm{Weil}^*(X)$ be a Weil form. Let η be as in the conclusion of* 2.3.3. *Then there exists a unique extension of η to a positive map $\eta\colon \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) \to \mathbf{R}$ such that*

(i) $-\log \eta$ *is bilinear;*

(ii) *for any $Z \in \mathbf{Q}ZX(K_a)$, the map $\eta(\cdot, Z)\colon X(K) \to \mathbf{R}$ is continuous;*

(iii) *for any $D \in \mathbf{Q}\mathrm{Div}(X)$, $-\log \eta(D, \cdot) : X(K_a) \to \mathbf{R}$ is continuous and bounded.*

Moreover, this extension is symmetric on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X)$.

Proof. We proceed in several steps:

Step 1. Construction of the extension. For any $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$, let ϕ_D be the choice of Weil function corresponding to the function η in the conclusion of 2.3.3 and let η_D be as in 2.3.2(b). Thus η_D is a continuous function on $X(K_a)$ with bounded logarithm, and $\eta(D, D') = \eta_D(D')$ for $D' \in \mathbf{Q}\mathrm{CDiv}_i(X)$. More generally, define $\eta(D, D') = \eta_D(D')$ for all $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$ and $D' \in \mathbf{Q}\mathrm{Div}(X)$ (where the right-hand side was defined prior to 2.3.3). For every $D \in \mathrm{Div}(X)$ we will define a function $\bar{\eta}_D\colon X(K_a) \to \mathbf{R}$ in such a way that the induced linear map $-\log \bar{\eta}\colon \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X) \to \mathbf{R}$ given by $(D, D') \mapsto -\log \bar{\eta}_D(D')$ extends $-\log \eta$.

Let $D \in \mathrm{CDiv}_i(X)$, so that D is the divisor defined by a homogeneous form F_D in $z_0, \ldots, z_n$ of degree $d \geqslant 0$, and consider the linear system $|D|$, which is a projective space over K. We claim that the function $|D| \times X(K_a) \to \mathbf{R}$ given by $(E, P) \mapsto -\log \eta_E(P)$ is bounded. To see this, first observe that the function $\phi_D + \log |F_D|$ extends to a bounded continuous function on $X(K_a)$, because ϕ_E is a Weil function. Also, if $E \in |D|$ then E is defined by a homogeneous form F_E of degree d, which may be chosen so that $|F_E|\,(D) = |F_D|\,(E)$. We then have

$$\eta_E = \eta_D \, |F_E/F_D| \, / \rho_{(F_E/F_D)} = (\Phi_D \, / \, |F_D|)(|F_E| \, / \rho_E),$$

where $\Phi_D = \exp(-\phi_D)$. Now $-\log(\Phi_D/|F_D|)$ is bounded on $X(K_a)$, since ϕ_D and $-\log |F_D|$ are Weil functions for D. By the proof of 2.3.2(b), the function $-\log |F_E| \, / \rho_E$ is bounded on $X(K_a)$, and its bound varies continuously in F_E, where F_E is viewed as belonging to the K-vector space of homogeneous forms over K in $z_0, \ldots, z_n$ of degree d. So for each $E \in |D|$, $\sup(-\log \eta_E)$ and $\inf(-\log \eta_E)$ exist, and these vary continuously as E ranges over the K-projective space $|D|$. By the compactness of this projective space, the claim follows.

Now let $D \in \mathrm{Div}(X)$ be an effective divisor of degree $s \geqslant 1$. Let $D(0) = hD$, where $h = g+1 \geqslant 1$ and $g = \text{genus } X$. So $D(0)$ is of degree hs, and the degree of $3shH - eD(0)$ is $hs > g$, where H is a hyperplane section defined by a linear form L, $e = 3d - 1$, and $d =$ degree of X in $\mathbf{P}^n$. So by the Riemann–Roch theorem over arbitrary fields [Chev, Chap. II], the corresponding complete linear system $|3shH - eD(0)|$ is non-empty. So we may inductively define a sequence of effective divisors $D(r)$ of degree hs such that $D(r) \in |3shH - eD(r-1)|$. For each $r \geqslant 0$, let $E(r) = eD(r) + D(r+1)$. So $E(r) \in |3shH|$. Thus by the previous paragraph there is a constant $c \geqslant 1$ such that for all $E \in |3shH|$ and $P \in X(K_a)$, $c^{-1} \leqslant \eta_E(P) \leqslant c$. In particular, $c^{-1} \leqslant \eta_{E(r)}(P) \leqslant c$. Let

$$F(r) = e^r D_0 + (-1)^{r-1} D(r) = \sum_{j=0}^{r-1} (-1)^j e^{r-1-j} E(j).$$

Then $\eta_{F(r+1)}/\eta^e_{E(r)} = \eta_{E(r)}^{(-1)^r}$, which takes on values between c^{-1} and c. Thus the values of $\eta_{F(r+1)}^{1/he^{r+1}}/\eta_{F(r)}^{1/he^r}$ lie between $c^{-1/he^{r+1}}$ and $c^{1/he^{r+1}}$. So the rth and $(r+1)$th terms in the sequence $\{(1/he^r) \log \eta_{F(r)}\}$ differ under the uniform norm by at most $(1/he^{r+1}) \log c$. Since $e > 1$, this implies that the sequence is Cauchy under the uniform norm. So $\lim_{r \to \infty} \eta_{F(r)}^{1/he^r}$ exists; denote it by $\bar{\eta}_D$.

It is straightforward to verify that $\bar{\eta}_D$ is independent of the choice of the divisors $D(r)$, that $\bar{\eta}_D \colon X(K) \to \mathbf{R}_+$ is continuous and has bounded logarithm, and that for $Z \in X(K_a)$ the function $D \mapsto \bar{\eta}_D$ varies continuously as a function of $D \in X(K_a)$. Extending the association $D \mapsto -\log \bar{\eta}_D$ by linearity in D, we obtain functions $\bar{\eta}_D$ for all $D \in \mathbf{Q}\mathrm{Div}(X)$. By linearity, each $\bar{\eta}_D$ induces a map $\bar{\eta}_D \colon \mathbf{Q}ZX(K_a) \to \mathbf{R}$, and we obtain a map $\bar{\eta} \colon \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) \to \mathbf{R}$ which satisfies properties (i)–(iii).

Step 2. Verification that $\bar{\eta}$ extends η; i.e., that $\bar{\eta}_D = \eta_D$ for all complete intersection divisors D. Again we may assume that D is effective, say $D = \mathrm{div}(G)$, where G is homogeneous of degree t. So $s = \deg D = td$. In the notation of Step 1, if $r > 0$ is odd then $F(r)$ is effective,

$$\begin{aligned} F(r) &= e^r D(0) + D(r) = \sum_{j=0}^{r-1} (-1)^j e^{r-1-j} E(j) \\ &\in \left| 3sh \sum_{j=0}^{r-1} (-1)^j e^{r-1-j} H \right| \\ &= |th(e^r + 1)H| \end{aligned}$$

(using $3s = 3td = t(e+1)$), and $D \in |tH|$. Thus

$$3dF(r) - 3dhe^rD \in |3shH|$$

and so $\eta_{(3d \cdot F(r) - 3dhe^rD)}$ takes on values between c^{-1} and c. So the values taken on by $\eta_D^{-1}\eta_{F(r)}^{1/he^r}$ range between $c^{-1/3dhe^r}$ and $c^{1/3dhe^r}$, and these bounds approach 1 as $r \to \infty$. But $\lim_{r \to \infty} \eta_F^{1/he^r} = \bar{\eta}_D$. So $\bar{\eta}_D = \eta_D$, as desired.

Thus we may henceforth write η_D for $\bar{\eta}_D$ for *all* $D \in \mathbf{Q}\mathrm{Div}(X)$, and write η for $\bar{\eta}$.

Step 3. Verification that the η constructed above is the unique extension of the given η which satisfies (i)–(iii). Suppose η' also extends the original η which was given on complete intersection divisors. Given an effective $D \in \mathrm{Div}(X)$, we wish to show $\eta'_D = \eta_D$. By the boundedness assumption (iii), there exists $c' \geqslant 1$ such that for all $P \in X(K_a)$, we have $c'^{-1} \leqslant \eta'_D(P) \leqslant c'$. By the claim in Step 1, there exists a constant $c'' \geqslant 1$ such that for all $E \in |D|$ and all $P \in X(K_a)$, $c''^{-1} \leqslant \eta_E(P) \leqslant c''$. Now if $E \in |D|$ then $E - D$ is linearly equivalent to 0, and hence $E - D \in \mathrm{CDiv}_i(X)$. Thus $\eta'_{E-D} = \eta_{E-D}$, and so $\eta'_E = \eta_E \eta'_D / \eta_D$. So for $E \in |D|$ and $P \in X(K_a)$, $k^{-1} \leqslant \eta'_E(P) \leqslant k$, where $k = c''^2 c'$. So as in Step 1, there are sequences of divisors $D(r)$, $E(r)$, $F(r)$. By the bounds on $\eta'_D(P)$, for all $P \in X(K_a)$ we have

$$\begin{aligned}
\eta'_D(P) &= \lim_{r \to \infty} (\eta'_D(P)^{e^r} \eta'_{D(r)}(P)^{(-1)^{r-1}})^{1/e^r} \\
&= \lim_{r \to \infty} (\eta'_{F(r)}(P))^{1/e^r} \qquad \text{(by definition of } F(r)) \\
&= \lim_{r \to \infty} \eta_{F(r)}(P)^{1/e^r} \qquad \text{(since } \eta' \text{ extends the original } \eta) \\
&= \eta_D(P) \qquad \text{as desired; so } \eta' = \eta.
\end{aligned}$$

Step 4. η is symmetric. That is, if D, $D' \in \mathbf{Q}\mathrm{Div}(X)$, then

$$\eta_D(D') = \eta'_D(D).$$

It suffices to consider effective divisors D, $D' \in \mathrm{Div}(X)$ having the same degree s. Thus in the notation of Step 1, $\eta_D = \lim_{r \to \infty} \eta_{F(r)}^{1/he^r}$, where $F(r) = e^rhD + (-1)^{r-1} D(r)$. Similarly, the construction of $\eta_{D'}$ gives $\eta_{D'} = \lim_{r \to \infty} \eta_{F'(r)}^{1/he^r}$, where $F'(r) = e^rhD' + (-1)^{r-1} D'(r)$.

As seen in Step 1, the function $\eta_{E(r)}\colon X(K) \to \mathbf{R}$ takes on values between c^{-1} and c. So $\eta_{F(r)} = \prod_{j=0}^{r-1} \eta_{E(j)}^{(-1)^j e^{r-1-j}}$ takes on values between $c^{-\Sigma e^{r-1-j}} = c^{-(e^r-1)/(e-1)}$ and $c^{(e^r-1)/(e-1)}$, on $X(K)$. Since $\deg D'(r) = hs$, $\eta_{F(r)}(D'(r))$

lies between $(c^{hs(e^r-1)/(e-1)})^{\pm 1}$. These quantities approach 1 as $r \to \infty$, and so $\eta_{(F(r)}(D'(r))^{(-1)^{r-1}/(he^r)^2} \to 1$ as $r \to \infty$. Thus,

$$\begin{aligned}\eta_D(D') &= \lim_{r \to \infty} \eta_{F(r)}(D')^{1/he^r} \\ &= \lim_{r \to \infty} [\eta_{F(r)}(F'(r))^{1/(he^r)^2} \eta(F(r))(D'(r))^{(-1)^{r-1}/(he^r)^2}] \\ &= \lim_{r \to \infty} \eta_{F(r)}(F'(r))^{1/(he^r)^2}.\end{aligned}$$

By symmetry, we have $\eta_{D'}(D) = \lim_{r \to \infty} \eta_{F'(r)}(F(r))^{1/(he^r)^2}$. But $\eta_{F(r)}(F'(r)) = \eta_{F'(r)}(F(r))$, since (by the previous lemma) η is symmetric on complete intersection divisors. Thus $\eta_D(D') = \eta_{D'}(D)$, as desired. ∎

If X is a smooth projective curve over a local field K, define a *Green's pairing* on X to be a real-valued bilinear function $\langle\ ,\ \rangle$ on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta$ (where Δ = pairs with non-disjoint support) such that

(i) for each $D \in \mathbf{Q}\mathrm{Div}(X)$ the function $P \mapsto \langle D, P\rangle : X(K_a) - \operatorname{supp} D \to \mathbf{R}$ is a Weil function for D;

(ii) if $D = (f) \in \mathrm{Div}(X)$ is linearly equivalent to 0, then there is a $c \in \mathbf{R}$ such that $\langle D, P\rangle = -\log |f|\ (P) + c$;

(iii) $\langle\ ,\ \rangle$ is symmetric on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X) - \Delta$.

Equivalently, to give a Green's pairing $\langle\ ,\ \rangle$ is to give a homomorphism $g: \mathbf{Q}\mathrm{Div}(X) \to \mathbf{Q}\mathrm{Weil}(X)$ such that the induced map ϕ: $\mathbf{Q}\mathrm{Div}(X) \to \mathbf{Q}\mathrm{Weil}^*(X)$ is a Weil form, and such that $(g(D))(D') = (g(D'))(D)$ for $D, D' \in \mathbf{Q}\mathrm{Div}(X)$. Here $\langle D, D'\rangle = (g(D))(D')$, and if $\langle\ ,\ \rangle$ is related to g in this manner we will also write $g(D, D')$ for $\langle D, D'\rangle$, and call the Green's pairing g. Note that if g is a Green's pairing and $c \in \mathbf{R}$, then a new Green's pairing $g + c$ may be defined by $(g+c)(D, D') = g(D, D') + c(\deg D)(\deg D')$. Thus, for example, if $P, Q \in X(K)$ then $(g+c)(P, Q) = g(P, Q) + c$. Also, if g induces the Weil form ϕ, then so does $g + c$, for any constant c.

We now state the key result of this section, which can be regarded as an analog of Proposition 1.3.1 for projective curves over arbitrary local fields. As before, given an embedding $i: X \hookrightarrow \mathbf{P}^n_K$ of a curve into projective space, we write $\mathrm{CDiv}_i(X) = i^*(\mathrm{Div}(\mathbf{P}^n_K))$ and $\mathbf{Q}\mathrm{CDiv}_i(X) = \mathrm{CDiv}_i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

2.3.5. THEOREM. *Let $i: X \hookrightarrow \mathbf{P}^n_K$ be an embedding of a smooth projective curve over a local field K, and let $\phi: \mathbf{Q}\mathrm{CDiv}_i(X) \to \mathbf{Q}\mathrm{Weil}^*(X)$ be a Weil form. Then there exists a Green's pairing on X which induces ϕ, and this pairing is unique up to adding a constant.*

First Proof. Let $\eta\colon \mathbf{Q}\mathrm{Div}(X)\times \mathbf{Q}ZX(K_a)\to \mathbf{R}$ be as in the conclusion of 2.3.4. For $(A, B)\in \mathbf{Q}\mathrm{Div}(X)\times \mathbf{Q}ZX(K_a)-\Delta$, let $G(A, B)=\rho(A, B)\,\eta(A, B)$, where $\rho(A, B)=\rho_A(B)$. Let $g=-\log G\colon \mathbf{Q}\mathrm{Div}(X)\times \mathbf{Q}ZX(K_a)-\Delta\to \mathbf{R}$. Then g is symmetric on $\mathbf{Q}\mathrm{Div}(X)\times \mathbf{Q}\mathrm{Div}(X)-\Delta$, because ρ and η are, and g is bilinear because $-\log\rho$ and $-\log\eta$ are. Also, for each $D\in \mathbf{Q}\mathrm{Div}(X)$, the function $g(D,\cdot)$ is a Weil function because $-\log\eta(D,\cdot)$ is a continuous bounded function and because ρ_D is a Weil function (2.3.2(a)). Moreover, if $D\in \mathbf{Q}\mathrm{CDiv}_i(X)$ then $g(D,\cdot)=-\log\rho_D+\eta_D$ is a choice of ϕ_D, by definition of η_D in 2.3.3. So g induces ϕ. In particular, given a divisor $D=(f)$ which is linearly equivalent to 0, we have $D\in \mathbf{Q}\mathrm{CDiv}_i(X)$, and so $g(D,\cdot)=\phi_D=-\log|f|+c$, for some $c\in\mathbf{R}$. Thus g is a Green's pairing which induces ϕ. Finally, uniqueness for g up to a constant follows from the uniqueness for η in 2.3.4. ∎

As a result, we will obtain a new construction of the Néron pairing, as well as an extension of this pairing to one on divisors of arbitrary degree in a way that agrees over $\mathbf{C}$ with the projective Green's pairing of Section 1. Recall (Proposition 2.2.3) that for a local field K, Green's functions on $\mathbf{P}^n_K$ define a Weil form on $\mathbf{Q}\,\mathrm{Div}(\mathbf{P}^n_K)$, and this pulls back to a Weil form ϕ on $\mathbf{Q}\mathrm{CDiv}_i(X)$ for any embedding $i\colon X\hookrightarrow \mathbf{P}^n_K$ (Proposition 2.2.6). If X is a curve, then Theorem 2.3.5 yields a Green's pairing g on X which induces ϕ. We call such a Green's pairing a *projective* Green's pairing on X relative to i; it is unique up to a constant. For each $D\in\mathbf{Q}\mathrm{Div}(X)$ we call the function $P\mapsto g(D, P)$ a *Green's function* for D on X relative to i. This agrees with the previous terminology in the case that $D\in\mathbf{Q}\mathrm{CDiv}_i(X)$, and for all D this is a Weil function for D on X.

2.3.6. THEOREM. *Let X be a smooth projective curve over a local field K, and let g be a Green's pairing on X (e.g., a projective one).*

(a) *The restriction of g to* $\mathrm{Div}_0(X)\times Z_0X(K_a)-\Delta$ *is the* Néron height pairing *on X, i.e., the unique bilinear pairing which is symmetric on* $\mathrm{Div}_0(X)\times\mathrm{Div}_0(X)-\Delta$, *such that if $f\in K(X)$ and $z\in Z_0X(K_a)$ then* $\langle (f), z\rangle=-\log|f(z)|$ *and such that if* $D\in\mathrm{Div}_0(X)$ *and* $P\in X(K_a)-\mathrm{supp}(D)$ *then $Q\mapsto\langle D, Q-P\rangle$ is continuous and locally bounded on* $X(K_a)-\{P\}$.

(b) *If $K=\mathbf{C}$ and g is a projective Green's pairing induced by $i\colon X\hookrightarrow\mathbf{P}^n_{\mathbf{C}}$, then g is a Green's pairing on the Riemann surface $X(\mathbf{C})$ relative to the pullback of the Fubini–Study metric on $\mathbf{P}^n(\mathbf{C})$ (in the sense of Section* 1.3).

Proof. (a) That the restriction has the given properties follows from the definition of a Green's pairing. To see that the pairing on $\mathrm{Div}_0(X)\times$

$Z_0X(K_a)-\Delta$ is unique, we follow the proof of [La, Chap. 11, Theorem 3.6]. Namely, if ϕ and ϕ' are two such pairings, then $h=\phi-\phi'$ is a symmetric bilinear pairing which is trivial if the first entry (or, by symmetry, the second) is a principal divisor. Also, if $D\in \mathrm{Div}_0(X)$ and $P\in X(K_a)-\mathrm{supp}(D)$ then $Q\mapsto \phi(D, Q-P)$ has logarithmic singularities along D, and similarly for ϕ'. So $Q\mapsto h(D, Q-P)$ extends to a continuous function on all of $X(K_a)$, which is locally bounded, and hence bounded (since X is projective), say by $M_P>0$.

So given $(A, B)\in \mathrm{Div}_0(X)\times Z_0X(K_a)-\Delta$, pick $P\in X(K_a)$ and let L be a finite extension of K such that $B, P\in ZX(L)$. For every $m>0$ use Riemann–Roch on X_L to write $mB+gP=C+(f)$, where $C\in \mathrm{Div}(X_L)$ is effective of degree $g=\mathrm{genus}(X)$, and $f\in K(X)$. Then $mh(A, B)=h(A, mB)=h(A, C-gP)$, which is bounded independently of m (viz. by gM_P). Since this is true for all m, $h(A, B)=0$.

(b) This follows by the corresponding fact about Green's functions associated to complete intersection divisors (Proposition 2.2.6) and the uniqueness of both types of Green's pairings (Proposition 1.3.1 and Theorem 2.3.5). ∎

Remark. A Green's pairing for a curve X relative to a projective embedding actually arises from a K-valued pairing on the Hopf bundle over X. Namely, as in Remark (c) after Proposition 2.2.6, Green's functions $|F|$ for complete intersection divisors lift to K-valued functions F on the Hopf bundle over X (relative to the projective embedding). Using these in place of Green's functions $|F|$, the results 2.3.1–2.3.5 carry over to results about K-valued functions and pairings on the Hopf bundle. For example, in the case $K=\mathbf{Q}_p$, the Green's pairing in 2.3.5 is the absolute value of a p-adic valued pairing on the Hopf bundle, and we may regard this as a p-adic valued Green's pairing for X. Restricting to degree 0, this gives a p-adic height pairing.

Alternatively, assuming as known the existence and uniqueness of the Néron pairing on degree 0 divisors, we obtain another, simpler proof of Theorem 2.3.5 (which relies on 2.3.1 but not on 2.3.2–2.3.4), as well as the fact that the Green's pairing extends the Néron pairing:

Second Proof of 2.3.5. Choose a hyperplane section H of X, and choose any Weil function g_H lying over $\phi(H)\in \mathrm{Weil}^*(X)$. For any $D\in \mathbf{Q}\mathrm{CDiv}_i(X)$, lift $\phi(D)$ to the unique $\mathbf{Q}$-Weil function ϕ_D for D such that $\phi_D(H)=\phi_H(D)$. We claim that $\phi_D(D')=\phi_{D'}(D)$ for $D, D'\in \mathbf{Q}\mathrm{CDiv}_i(X)$ having disjoint support. Namely, by Lemma 2.3.1, if the supports of D and D' are disjoint from that of H then $\phi_D(D')+\phi_{D'}(H)+\phi_H(D)=\phi_{D'}(D)+\phi_D(H)+\phi_H(D')$. So indeed $\phi_D(D')=\phi_{D'}(D)$ for such D, D', and by continuity the claim follows.

Let $\sigma: \mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a) - \Delta$ be the bilinear extension of the Néron height pairing to a pairing over $\mathbf{Q}$, and for $(A, B) \in \mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a) - \Delta$ let $\sigma_A(B) = \sigma(A, B)$. If $D \in \mathrm{CDiv}_i(X)$ has degree 0, then D is linearly equivalent to 0; if $D = (f)$ then as a function on $X(K_a)$ we have $\phi_D = -\log |f| + c$ for some $c \in \mathbf{R}$. So $\phi_D = \sigma_D$ on $\mathbf{Q}Z_0X(K_a)$ for such D, and hence on $\mathbf{Q}\mathrm{CDiv}_i(X) \cap \mathbf{Q}\mathrm{Div}_0(X)$. Also, by the symmetry of ϕ, if $(A, B) \in \mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a) - \Delta$, where $B = (f)$ is in fact a principal divisor on X, then $\phi_A(B) = -\log |f|\ (A)$.

Now every $D \in \mathbf{Q}\mathrm{Div}(X)$ may be written as $D = D' + D''$, where $D' \in \mathbf{Q}\mathrm{Div}_0(X)$ and $D'' \in \mathbf{Q}\mathrm{CDiv}_i(X)$. Similarly, if $E \in \mathbf{Q}ZX(K_a)$ we may write $E = E' + E''$, where $E' \in \mathbf{Q}Z_0X(K_a)$ and $E'' \in \mathbf{Q}\mathrm{CDiv}_i(X)$. So for such D, E with disjoint support, define

$$g(D, E) = \sigma(D', E') + \phi_{D''}(E') + \phi_{E''}(D') + \phi_{D''}(E'').$$

This is well defined, since $\phi_D = \sigma_D$ on $\mathbf{Q}\mathrm{CDiv}_i(X) \cap \mathbf{Q}\mathrm{Div}_0(X)$. Also, g_D agrees with ϕ_D on $\mathbf{Q}\mathrm{CDiv}_i(X)$, and g agrees with σ on $\mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a) - \Delta$. It remains to check that g is a Green's pairing and is unique up to a constant.

To show that g is a Green's pairing we first need to verify that for each $D \in \mathbf{Q}\mathrm{Div}(X)$ the function $g_D: z \mapsto g(D, z)$ is a Weil function for D. Now $g_D(E) = g_{D''}(E) + \gamma_{D'}(E)$, where $\gamma_{D'}(E) = \sigma(D', E') + \phi_{E''}(D')$ and where the decompositions $D = D' + D''$ and $E = E' + E''$ are as above. Since $g_{D''}$ is a Weil function for D'', it suffices to show that $\gamma_{D'}$ is a Weil function for D'. Let $C \in \mathbf{Q}\mathrm{Div}(X)$ be a $\mathbf{Q}$-divisor linearly equivalent to D' and having support disjoint from that of D'. Thus $D' - C = (f)$ for some $f \in K(X)$, and f is a local equation for D'. For any $\mathbf{Q}$-divisor E' having support disjoint from those of C and D', $\gamma_{D'}(E) + \log |f|\ (E) = [\sigma(C, E') - \log |f|\ (E')] + [\phi_{E''}(C) - \log |f|\ (E'')] + \log |f|\ (E) = \sigma(C, E') + \phi_{E''}(C)$, which extends to a continuous locally bounded function on the complement of the support of C. So indeed $\gamma_{D'}$, and hence g_D, is a Weil function. Next, if $D = (f)$ is linearly equivalent to 0, then $D \in \mathrm{Div}_i(X)$, and for $P \in X(K_a)$ we have $g_D(E) = \phi_D(P) = -\log |f|\ (P) + c$ for some c. Moreover, g is symmetric because $\sigma(D', E') = \sigma(E', D')$ and $\phi_{D''}(E'') = \phi_{E''}(D'')$.

Finally, we show uniqueness up to a constant. Suppose that g' is another such Green's pairing. Then for each $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$, the Weil function g'_D induces ϕ_D, and hence differs from g_D by a constant $c_D \in \mathbf{R}$, as a function on $X(K_a)$. So with H as above, $g'(H, D) = g'(D, H) = g(D, H) + c_D(\deg H) = g(H, D) + c_D(\deg H)$, and so $c_D(\deg H) = c_H(\deg D)$ for all D. Thus there is a constant $c = c_H/(\deg H)$ such that $g'_D = g_D + c(\deg D)$ as a function on $X(K_a)$, for all $D \in \mathbf{Q}\mathrm{CDiv}_i(X)$. Thus $g'_D = g_D$ if $\deg D = 0$, and more generally $g'(D, E) = g(D, E) + c(\deg D)(\deg E)$. Also, the restriction of g' to $\mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a) - \Delta$ satisfies all the properties of the Néron height pairing, and so it must agree there with g.

Now as before, each $D \in \mathbf{Q}\mathrm{Div}(X)$ and each $E \in \mathbf{Q}ZX(K_a)$ may be respectively decomposed as $D = D' + D''$ and $E = E' + E''$, where the first term has degree 0 and the second lies in $\mathbf{Q}\mathrm{CDiv}_i(X)$. If D and E have disjoint support we have $g'(D, E) = g'(D', E') + g'(D', E'') + g'(D'', E) = g(D', E') + g(D', E'') + g(D'', E) + c(\deg D'')(\deg E) = g(D, E) + c(\deg D) \cdot (\deg E)$. That is, $g' = g + c$. ▌

2.3.7. PROPOSITION. *Let X be a curve over a local field K, and let $g, g' \colon \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta \to \mathbf{R}$ be Green's pairings on X. Then there is a unique linear transition function $\tau \colon \mathbf{Q}ZX(K_a) \to \mathbf{R}$ which is continuous on $X(K_a)$, such that if $(A, B) \in \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta$ then*

$$g'(A, B) = g(A, B) + d_B\tau(A) + d_A\tau(B), \tag{*}$$

where d_A and d_B are the degrees of A and B. In particular, if P, Q are distinct K-points of X then $g'(P, Q) = g(P, Q) + \tau(P) + \tau(Q)$.

Conversely, suppose that g is a Green's pairing on X and suppose that $\tau \colon \mathbf{Q}ZX(K_a) \to \mathbf{R}$ is a linear function which is continuous and bounded on X. Then the pairing g' defined by $(*)$ is a Green's pairing.

Proof. Let $h = g' - g$. Since g_D and g'_D are Weil functions for each $D \in \mathbf{Q}\mathrm{Div}(X)$, $h_D = g'_D - g_D$ extends to a bounded continuous function on all of $X(K_a)$, and h extends to a bilinear pairing on all of $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a)$. The pairing h is trivial on pairs of divisors $(A, B) \in \mathbf{Q}\,\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a)$ of degree 0, since g and g' both agree with the Néron pairing. Also, h is symmetric on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X)$, since g and g' are.

If $A \in \mathbf{Q}Z_0X(K_a)$ is a $\mathbf{Q}$-zero cycle of degree 0 and $B, C \in \mathbf{Q}\mathrm{Div}(X)$ have the same degree, then $h(B - C, A) = 0$. So the function $B \mapsto h(B, A)$ is constant on the space of $\mathbf{Q}$-divisors B of any fixed degree. Thus by the bilinearity of h, there exists a constant $\tau(A) \in \mathbf{R}$ such that for all $B \in \mathbf{Q}\mathrm{Div}(X)$, $h(B, A) = d_B\tau(A)$, where d_B is the degree of B. Also, $\tau \colon \mathbf{Q}Z_0X(K_a) \to \mathbf{R}$ is linear, and τ is continuous on $X(K_a)$, because h_B has these properties for all $B \in \mathbf{Q}\mathrm{Div}(X)$.

Now fix some $\mathbf{Q}$-divisor $D \in \mathbf{Q}\mathrm{Div}(X)$ of degree 1. For any $\mathbf{Q}$-divisor $A \in \mathbf{Q}\mathrm{Div}(X)$, say of degree d_A, $h(B, A - d_A D) = d_B\tau(A - d_A D)$ for all $B \in \mathbf{Q}\mathrm{Div}(X)$ of degree d_B. Using bilinearity and the symmetry of h on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X)$, $d_B\tau(A - d_A D) + d_A h(B, D) = h(B, A) = h(A, B) = d_A\tau(B - d_B D) + d_B h(A, D)$. So if d_A and d_B are non-zero, then $[\tau(A - d_A D) - h(A, D)]/d_A = [\tau(B - d_B D) - h(D, B)]/d_B$. Thus $[\tau(A - d_A D) - h(A, D)]/d_A$ is independent of A, say equal to $c \in \mathbf{R}$. So $h(A, D) = \tau(A - d_A D) - c d_A$, and this is true even if $d_A = 0$ (since $h(D, A) = \tau(A)$ for A of degree 0, as shown above). Thus $h(A, B) = d_B h(A, B) + h(A, B - d_B D) = d_B\tau(A - d_A D) - c\, d_A d_B + d_A\tau(B - d_B D)$.

For arbitrary $A \in \mathbf{Q}ZX(K_a)$ of degree d_A, let $\tau(A) = \tau(A - d_A D) - cd_A/2$; this agrees with the previous definition of $\tau(A)$ if $d_A = 0$. Thus $\tau: \mathbf{Q}ZX(K_a) \to \mathbf{R}$ is linear, and τ is bounded and continuous on $X(K_a)$. Finally, suppose that $(A, B) \in \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta$, with A, B of degrees d_A, d_B. Then $g'(A, B) - g(A, B) = h(A, B) = h(A, d_B D) + h(A, B - d_B D) = d_B \tau(A - d_A D) - c\, d_A d_B + d_A \tau(0) + d_A \tau(B - d_B D) = d_B \tau(A) + d_A \tau(B)$, as desired.

For uniqueness, suppose τ' is also such a transition function. Let $\sigma = \tau' - \tau$. We wish to show that $\sigma(A) = 0$ for all $A \in \mathbf{Q}\mathrm{Div}(X)$. So given an A, pick non-zero effective divisors B, C on X such that A, B, C have pairwise disjoint support. Then $d_A \sigma(C) + d_C \sigma(A) = 0$ and $d_B \sigma(C) + d_C \sigma(B) = 0$, and so $d_B d_C \sigma(A) = d_A d_C \sigma(B)$. Since d_B, $d_C \neq 0$ and since $d_A \sigma(B) + d_B \sigma(A) = 0$, we have $d_A \sigma(B) = d_B \sigma(A) = 0$, and so indeed $\sigma(A) = 0$.

The converse is immediate from the definition of Green's pairings. ▮

Remarks. (a) Proposition 2.3.7 is analogous to Arakelov's result [Ar, Proposition 3.2] over $\mathbf{C}$; cf. Remark (b) after 1.3.1 above.

(b) Kani has defined a notion of "v-height," closely related to the notion of Green's pairing. He showed the existence of v-heights induced by morphisms to $\mathbf{P}^1_K$ and proved the analog of Proposition 2.3.7. See [Ka].

2.3.8. Example. Let X be a smooth curve of degree d in $\mathbf{P}^2_K$ and let $P \in X(K)$. Let Y be the blowup of $\mathbf{P}^2$ at P, so that $Y \subset \mathbf{P}^2 \times \mathbf{P}^1$, which we embed in $\mathbf{P}^5$ by the Segre map. Thus we have two projective embeddings of X: the given one $i: X \hookrightarrow P^2$, and the embedding $i': X \hookrightarrow P^5$ via blowup and Segre. Let g and g' be projective Green's pairings with respect to the embeddings i and i' (each being determined uniquely up to a constant). Then after adjusting g' by a constant, the transition function in 2.3.7 is given by

$$\tau(Q) = [g(P, Q) + \log \rho(P, Q)]/(2d - 1).$$

To see this, it suffices (by 2.3.5) to verify (*) of 2.3.7 in the case that A is a complete intersection divisor (F) relative to the embedding i. With respect to the embedding i, a projective Green's function for A is given by $g_A(Q) = -\log |F|\,(Q)$, and one verifies explicitly that a projective Green's function for A with respect to i' is given by $g'_A(Q) = \tau(Q) + g_A(Q)$, where τ is as above. It then follows that τ must be the transition function (up to adding a constant).

For example, let $K = \mathbf{C}$, let X be the conic $(x - z)^2 + y^2 = z^2$, and blowup at $\mathbf{P} = (0:0:1)$. Then the transition function is given by $\tau(a:b:1) = -(1/6) \log[|a|\,(|a|^2 + |b|^2 + 1)^{1/2}/(|a|^2 + |b|^2)]$.

2.3.9. COROLLARY. *Let X be a complex projective curve. Then the set of Green's pairings g on $X(\mathbf{C})$ relative to Hermitian metrics $d\mu$ on $X(\mathbf{C})$ (in the sense of Section* 1.3) *is the same as the set of Green's pairings on X (in the present sense).*

Proof. Arakelov [Ar, Proposition 3.2] proved the existence of transition functions f between any two Green's functions on a curve relative to two given Hermitian metrics (cf. Remark (b) after Proposition 1.3.1). In particular, if g is Green's function on X induced by the pullback of Fubini–Study under a projective embedding $X \subsetneq \mathbf{P}^n_{\mathbf{C}}$, and g' is some other bilinear pairing, then g' is a Green's function if and only if g and g' are related by such a transition function. So the result follows from Theorem 2.3.6(b) and Proposition 2.3.7. ∎

Replacing the projective Green's functions of 2.3.6 by Néron's pairing on the Jacobian, we will obtain from 2.3.5 another extension of the Néron height pairing to divisors of arbitrary degree, which over $\mathbf{C}$ is the Arakelov pairing with respect to the "canonical" metric.

Recall [La, Chap. 11, Theorem 2.2; Ne, Chap. 2, Section 7, Theorem 1] that for any abelian variety A over K there is a unique translation invariant Weil form. This Weil form, or the corresponding pairing $\mathbf{Q}\mathrm{Div}(A) \times \mathbf{Q}Z_0A(K_a) - \Delta \to \mathbf{R}$, is called the *Néron pairing* on the abelian variety A. In particular, let X be a projective curve over K of genus >0 and let A be the Jacobian of X. Let D be a divisor on X of degree 1; this must exist by Riemann–Roch. We obtain an Albanese map $\alpha: X \subsetneq A$ given on closed points by $Q \mapsto [Q - \deg(Q)D]$. Now let $\mathrm{Div}_a(A) = \{$divisors on A algebraically equivalent to $0\}$ and choose a very ample divisor Y on A. Let $\mathrm{Div}_Y(A)$ be the subgroup of $\mathrm{Div}(A)$ generated by Y and $\mathrm{Div}_a(A)$. Then $\alpha^*: \mathrm{Div}_Y(A)/\sim \to \mathrm{Pic}\, X$ is injective, since $\alpha^*: \mathrm{Pic}_0(A) \to \mathrm{Pic}_0(X)$ is an isomorphism and since every element of $\mathrm{Div}_Y(A)$ can uniquely be written in the form $mY + Z$, $Z \in \mathrm{Div}_0(A)$. By Proposition 2.2.4, the Weil form on $\mathrm{Div}_Y(A)$ induced by the Néron pairing on A pulls back to a Weil form ϕ on $\alpha^*(\mathbf{Q}\mathrm{Div}_Y(A)) \subset \mathbf{Q}\mathrm{Div}(X)$. In fact, $\alpha^*(\mathbf{Q}\mathrm{Div}_Y(A)) = \mathbf{Q}\mathrm{Div}(X)$, since every element of $\mathbf{Q}\mathrm{Div}(X)$ is of the form $a\alpha^*(Y) + Z$, where $a \in Q$ and $Z \in \mathrm{Div}_0(X)$. So ϕ is a Weil form on all of $\mathbf{Q}\mathrm{Div}(X)$, and we call it the *Néron form* on X with respect to the divisor Y on A. The restriction of ϕ to a Weil form on $\mathbf{Q}\mathrm{Div}_0(X)$ lifts to the Néron pairing on the curve X [La, Chap. 11, Theorems 3.5 and 3.6]. In fact, more is true, and we have this generalization of 1.3.3 to curves over local fields:

2.3.10. THEOREM. *Let X be a smooth projective curve of genus >0 over a local field K, and let ϕ be the Néron form on $\mathbf{Q}\mathrm{Div}(X)$ with respect to a very ample divisor Y on $A = \mathrm{Jac}(X)$. Then:*

(a) *There exists a Green's pairing on X which induces ϕ, and this pairing is unique up to adding a constant.*

(b) *This pairing induces the Néron pairing on X in degree* 0.

(c) *If $K=\mathbf{C}$ and $NS(A)=\mathbf{Z}$, then this Green's pairing agrees (up to a normalizing constant) with the canonical Green's pairing on the Riemann surface $X(\mathbf{C})$ (in the sense of* 1.3.3).

Proof. As above, let $\alpha: X \hookrightarrow A=\mathrm{Jac}(X)$ be the Albanese map with respect to some $D \in \mathrm{Div}(X)$ of degree 1, and let $j: A \hookrightarrow \mathbf{P}_K^n$ be a projective embedding corresponding to Y. Thus Y is a hyperplane section of A, relative to j, and $\mathbf{Q}\mathrm{CDiv}_j(A)$ consists of the $\mathbf{Q}$-divisors linearly equivalent to some multiple of Y. Let $i=j\circ\alpha: X \hookrightarrow \mathbf{P}_K^n$. Then $\mathbf{Q}\mathrm{CDiv}_i(X)$ consists of the divisors on X such that some multiple is linearly equivalent to a multiple of $\alpha^*(Y)$. By Theorem 2.3.5, there is a Green's pairing g on X which induces the Weil form $\phi_i: \mathbf{Q}\mathrm{CDiv}_i(X) \to \mathrm{Weil}^*(X)$, where ϕ_i is the restriction of ϕ. Also by 2.3.5, g is unique for this property, up to a constant. To prove (a) and (b) it remains to show that g induces ϕ on all of $\mathbf{Q}\mathrm{Div}(X)$. But since g is a Green's pairing, its restriction to $\mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}Z_0X(K_a)$ must equal Néron's pairing (Theorem 2.3.6(a)); i.e., g induces the restriction of ϕ to $\mathbf{Q}\mathrm{Div}_0(X)$. Since $\mathbf{Q}\mathrm{CDiv}_i(X)$ and $\mathbf{Q}\mathrm{Div}_0(X)$ generate $\mathbf{Q}\mathrm{Div}(X)$, g indeed induces ϕ on all of $\mathbf{Q}\mathrm{Div}(X)$.

According to Proposition 1.3.3, the canonical Green's pairing on $X(\mathbf{C})$ induces the pullback ϕ of the Néron pairing on $A(\mathbf{C})$. And by Corollary 2.3.9, the canonical pairing on $X(\mathbf{C})$ is a Green's pairing on X, in the sense of this present section. So part (c) follows by the uniqueness assertion in part (a). ∎

Remarks. (a) The condition $NS(A)=\mathbf{Z}$ holds generically, and shows in particular that the Néron pairing on an elliptic curve agrees with the canonical Green's pairing, for arbitrary pairs of disjoint divisors. For arbitrary curves X, Remark (b) after 1.3.3 suggests that if Y is a theta-divisor on A, then the conclusion of 2.3.10(c) might hold even without an assumption on $NS(A)$. It would be interesting to know if this is the case.

(b) Theorem 2.3.10(c) and Remark (a) also suggest that over an arbitrary local field K, if Y is a theta-divisor, then (at least in the case that $NS(A)=\mathbf{Z}$, and perhaps more generally) the induced Green's pairing on X should be regarded as analogous to the canonical Green's pairing at infinite places. Specifically, one may ask whether this pairing satisfies an adjunction formula, in the sense of [Ha, Section 4]. Also, Chinburg and Rumely [CR] have constructed pairings for curves over non-archimedean local fields which they regard as analogous to the canonical pairing at infinite places, and it would be interesting to see if those pairings are related to the pairings in 2.3.10 (or those in [Ha, Section 4]).

2.4. *Global Green's Pairings*

The results of this chapter can be generalized to the case of fields with a "proper set of absolute values," e.g., global fields. In this section we discuss such global Green's pairings, showing that they satisfy various properties analogous to intersection pairings, and that they induce real-valued height pairings.

Recall [La, Section 2.1] that an absolute value $|\ |$ on a field K is called *well behaved* if for every finite extension L of K we have $[L:K]=\sum_i [L_i^*:K^*]$, where K^* is the completion of K and the fields L_i^* are the completions of L with respect to the extensions of $|\ |$. An absolute value $|\ |$ is *proper* if it is non-trivial, well behaved, and (in the case that char $K=0$) extends one of the standard absolute values $|\ |_p$ on $\mathbf{Q}$ (p finite or infinite). A set M of absolute values on K is called *proper* if each absolute value in M is proper, the absolute values are independent, and for all $x \neq 0$ in K we have $|x|_v=1$ for all but finitely many $|\ |_v \in M$. In particular, if K is a local field then the set consisting of its absolute value is proper, and if K is a global field then any subset of $M_K=\{$absolute values $|\ |_v$ extending a standard absolute value on the prime field$\}$ is proper. For any M, one defines an *M-constant* to be a map $c: M \to \mathbf{R}$ such that for all but finitely many $v \in M$, $c_v=0$ [La, Section 10.1]. Thus in particular, if $a \in K$ and $a \neq 0$, then the map $D(a)=-\log|a|: M \to \mathbf{R}$ given by $v \mapsto v(a)=-\log|a|_v$ is an M-constant, and we call such an M-constant *principal.* We may identify the group of M-constants with the real vector space $\mathbf{R}^M$ spanned by the elements of M.

Let M be a proper set of absolute values on a field K. In this situation there is a notion of "Weil function" which generalizes the notion in Section 2.2. Namely, if X is a K-variety and D is a Cartier divisor on X, then $\phi: (X-\operatorname{supp} D) \times M \to \mathbf{R}$ is a *Weil function* for D on X if for every rational function h on X which is a local equation for D on an open set U, the function $\phi+\log|h|: (U-\operatorname{supp} D) \times M \to \mathbf{R}$ extends to a continuous locally bounded function on $U \times M$ (cf. [La, Chap. 10, Sections 1, 2]). Extending by linearity on points, a Weil function for D can be regarded as a function on the 0-cycles disjoint from the support of D.

As in the case of local fields, the Weil functions on X form a group $\operatorname{Weil}(X)$, and there is a canonical homomorphism $\operatorname{Weil}(X) \to \operatorname{Div}(X)$. Let $\operatorname{Weil}^*(X)=\operatorname{Weil}(X)/(M\text{-constants})$, the quotient by the subgroup of M-constants (which can be regarded as Weil functions for the trivial divisor). Thus there are induced maps $\pi: \operatorname{Weil}^*(X) \to \operatorname{Div}(X)$ and $\pi: \mathbf{Q}\operatorname{Weil}^*(X) \to \mathbf{Q}\operatorname{Div}(X)$, where $\mathbf{Q}\operatorname{Div}(X)=\operatorname{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, $\mathbf{Q}\operatorname{Weil}(X)=\operatorname{Weil}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, and $\mathbf{Q}\operatorname{Weil}^*(X)=\mathbf{Q}\operatorname{Weil}(X)/\mathbf{R}$. If Div' is any subgroup of

$\mathbf{Q}\mathrm{Div}(X)=\mathrm{Div}(X)\otimes_{\mathbf{Z}}\mathbf{Q}$, we say that a *Weil form* on Div′ is a homomorphism $\phi\colon \mathrm{Div}'\to \mathbf{Q}\mathrm{Weil}^*(X)$ which is a section of π over Div′, and such that if $D=(f)$ is a principal divisor then $\phi(D)=-\log|f|\colon (X-\mathrm{supp}\,D)\times M\to\mathbf{R}$ (modulo M-constants).

In particular, consider the case that $X=\mathbf{P}^n(K)$. If $F(z_0,\ldots,z_n)$ is a homogeneous polynomial over K, and $P\in\mathbf{P}^n(K)$, then for each $v\in M$ choose $A_v\in K^{n+1}$ lying over P and satisfying $|A_v|_v=1$, and as in Section 2.1 set $|F|_v(P)=|F(A_v)|_v$. This is independent of the choice of A_v, and we obtain a function $|F|\colon \mathbf{P}^n(K)\times M\to\mathbf{R}$. We call $|F|$ (resp. $g_F=-\log|F|\colon (\mathbf{P}^n(K)-\mathrm{supp}\,D)\times M\to\mathbf{R}$, given by $(P,v)\mapsto -\log|F|_v(P)$) a *multiplicative Green's function* (resp. a *Green's function*) on $\mathbf{P}^n_K$ for the divisor $D=(F)\subset\mathbf{P}^n_K$. Extending $g_D=g_F$ by linearity in F, we obtain for every divisor D in $\mathbf{P}^n_K$ a Weil function $g_D\colon (\mathbf{P}^n_K-\mathrm{supp}\,D)\times M\to\mathbf{R}$, and call it a *Green's function* for the divisor D. Note that g_D is uniquely determined up to an M-constant; viz. if $D=(F')-(F'')=(E')-(E'')$, then $F'E''=aE'F''$ for some non-zero $a\in K$, and the two choices of g_D (viz. $g_{F'}-g_{F''}$ and $g_{E'}-g_{E''}$) differ by the M-constant $|a|$. Similarly, if F', F'' are homogeneous polynomials (not necessarily of the same degree) and $D=\mathrm{div}(F')-\mathrm{div}(F'')$, then we obtain a *multiplicative Green's function* $|F'/F''|\colon (\mathbf{P}^n(K)-\mathrm{supp}\,D)\times M\to\mathbf{R}$, and this is unique up to multiplying by $e^c\colon v\mapsto\exp(c_v)$ for some M-constant c. Thus the assignment $D\mapsto g_D$ is a Weil form on Div $\mathbf{P}^n_K$, and by linearity it extends to $\mathbf{Q}$Div $\mathbf{P}^n_K$.

The notion of Green's function carries over to more general projective varieties. Namely, following Section 2.2, if $i\colon X\hookrightarrow\mathbf{P}^n_K$ is an embedding of K-varieties and $D\in\mathrm{CDiv}_i(X)=i^*\,\mathrm{Div}(\mathbf{P}^n_K)$ is a complete intersection Cartier divisor on X relative to i (so that $D=i^*H$, H a hypersurface in $\mathbf{P}^n_K$), we define a *Green's function* for D on X relative to i to be the restriction of a Green's function g_H for H to the variety X. Again, Green's functions are Weil functions, and the assignment $D\mapsto g_D$ is a Weil form on $\mathrm{CDiv}_i(X)$ which extends to $\mathbf{Q}\mathrm{CDiv}_i(X)$. For fixed X, i, and M, we denote by $\mathrm{CGr}_{i,M}(X)$ (or simply $\mathrm{CGr}_i(X)$ if M is understood) the group of Green's functions of elements of $\mathrm{CDiv}_i(X)$, so that $\mathbf{Q}\mathrm{CGr}_i(X)=\mathrm{CGr}_i(X)\otimes\mathbf{Q}$ is the group of Green's functions of elements of $\mathbf{Q}\mathrm{CDiv}_i(X)$.

In the special case that X is a smooth curve in $\mathbf{P}^n_K$, we may again define a notion of "Green's pairing." Namely, a *Green's pairing* on X is a collection $\{\langle\ ,\ \rangle_v : v\in M\}$ of real-valued bilinear functions on disjoint pairs $\mathbf{Q}\mathrm{Div}(X)\times\mathbf{Q}ZX(K_a)-\Delta$ (where K_a is, as usual, the algebraic closure of K) such that

(i) for each $D\in\mathbf{Q}\mathrm{Div}(X)$, the function $(P,v)\mapsto\langle D,P\rangle_v\colon (X(K_a)-\mathrm{supp}\,D)\times M\to\mathbf{R}$ is a Weil function for D;

(ii) if $D\in\mathrm{Div}(X)$ is linearly equivalent to 0, then for any $f\in K(X)$

with $D=(f)$, there is an M-constant c such that $\langle D, P\rangle_v = -\log |f|_v (P) + c_v$ for all $P \in X(K_a)$ and $v \in M$;

(iii) $\langle\ ,\ \rangle_v$ is symmetric on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X) - \Delta$, for all $v \in M$.

Equivalently, to give a Green's pairing $\langle\ ,\ \rangle$ on a K-curve X is to give a homomorphism $g\colon \mathbf{Q}\mathrm{Div}(X) \to \mathbf{Q}\mathrm{Weil}(X)$ such that the induced map $\phi\colon \mathbf{Q}\mathrm{Div}(X) \to \mathbf{Q}\mathrm{Weil}^*(X)$ is a Weil form and such that the pairing $\langle D, P\rangle_v = g(D)(P, v)$ satisfies (ii) and (iii). Writing $g_v(D, D') = (g(D))(D', v)$, g corresponds to $\langle\ ,\ \rangle$ if $g_v(D, D') = \langle D, D'\rangle_v$. If c is an M-constant, write $g+c$ for the Green's pairing given by $(g+c)_v\,(D, D') = g_v(D, D') + c_v(\deg D)(\deg D')$; this induces the same Weil form as g. Also, g and $g+c$ restrict to the same Weil form on $\mathbf{Q}\mathrm{Div}_0(X)$, the $\mathbf{Q}$-divisors of degree 0. As in the local case (Theorem 2.3.6(a)), Green's pairings extend the Néron pairing on $\mathbf{Q}\mathrm{Div}_0(X) \times \mathbf{Q}ZX_0(K_a) - \Delta$, since the restriction to that subgroup satisfies the properties which characterize the Néron pairing. Since the generalization of the "triple reciprocity" Lemma 2.3.1 is easily seen to hold, the second proof of Theorem 2.3.5 (given after the proof of 2.3.6) carries over, and we obtain:

2.4.1. THEOREM. *Let K be a field and let M be a proper set of absolute values for K. Let $i\colon X \hookrightarrow \mathbf{P}^n_K$ be an embedding of a smooth projective curve over K, and let $\phi\colon \mathbf{Q}\mathrm{CDiv}_i(X) \to \mathbf{Q}\mathrm{Weil}^*(X)$ be a Weil form. Then there exists a Green's pairing on X which induces ϕ, and this pairing is unique up to adding an M-constant.*

In particular, under the hypotheses of 2.4.1, there is a *projective Green's pairing* g which extends the Weil form that assigns to divisors their Green's functions. As in 2.4.1, this is unique up to an M-constant.

Similarly, Proposition 2.3.7 generalizes:

2.4.2. PROPOSITION. *Let K, M be as in* 2.4.1, *let X be a smooth projective K-curve, and let g be a Green's pairing on X. Then g' is also a Green's pairing on X if and only if there is a transition function $\tau\colon \mathbf{Q}ZX(K_a) \times M \to \mathbf{R}$ which is linear in the first variable, continuous and locally bounded on $X(K_a) \times M$, and satisfies the property that if $(A, B) \in \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta$ and $v \in M$, then*

$$g'_v(A, B) = g_v(A, B) + d_B\tau(A, v) + d_A\tau(B, v), \tag{*}$$

where d_A and d_B are the degrees of A and B.

Alternatively, we may regard τ as a function $\mathbf{Q}ZX(K_a) \to \mathbf{R}^M$.

Let K, M, X, g be as in 2.4.2. Then by (ii) in the definition of Green's pairings, for every non-zero rational function $f \in K(X)^*$ there is an M-constant $g(f) = \{g_v(f)\}_{v \in M}$ such that

$$g_v((f), Y) + g_v(f)(\deg Y) = -\log |f|_v (Y)$$

for all $v \in M$ and all divisors Y disjoint from the support of (f). Note that $f \mapsto g(f)$ is a homomorphism from $K(X)^*$ to the group of M-constants, and that each map $f \mapsto g_v(f)$ depends only on X, K, and v, and not on the set M. Also observe that if g' is another Green's pairing and τ is the transition function between g and g', then $g(f) = g'(f) + \tau(f)$. In particular, if g and g' differ by an M-constant, then $\tau(f) = 0$ and so $g'(f) = g(f)$. Thus $g(f)$ actually depends only on the Weil form ϕ corresponding to g, rather than on the specific choice of Green's pairing g lifting ϕ. Note also that if v is a complex place, and $d\mu_v$ is the Hermitian metric on X_v of volume 1 corresponding to the local Green's pairing g_v (cf. Corollary 2.3.9), then $g_v(f) = -\int_{X_v} \log |f|_v \, d\mu_v$.

Green's pairings satisfy a projection formula (compare [Ar, Proposition 5.2; Ka, Lemma 3, p. 429]):

2.4.3. Proposition. *Let K, M be as in 2.4.1. Let $\phi\colon X' \to X$ be a non-constant morphism of projective curves over K. Let g be a Green's pairing on X, over M.*

(i) *There is a unique Green's pairing $g' = \phi^*(g)$ on X' such that*

$$g'_v(\phi^* D, Z') = g_v(D, \phi_* Z') \tag{1}$$

and

$$g'_v(D', \phi^* Z) = g_v(\phi_* D', Z) \tag{2}$$

for all $v \in M$; all $D \in \mathbf{Q}\mathrm{Div}(X)$ and $Z' \in \mathbf{Q}ZX'(K_a)$ such that $\phi^* D$ and Z have disjoint support; and all $D' \in \mathbf{Q}\mathrm{Div}(X')$ and $Z \in \mathbf{Q}ZX(K_a)$ such that D' and $\phi^* Z$ have disjoint support.

(ii) *For $D \in \mathbf{Q}\mathrm{Div}(X)$ and $Z \in \mathbf{Q}ZX(K_a)$ having disjoint support, and all $v \in M$,*

$$g'_v(\phi^* D, \phi^* Z) = (\deg \phi)\, g_v(D, Z). \tag{3}$$

Proof. (i) We may write every $D' \in \mathbf{Q}\mathrm{Div}(X')$ as $D'_1 + D'_2$, where $D'_1 \in \mathbf{Q}\mathrm{Div}_0(X')$ and $D'_2 = \phi^*(D_2) \in \phi^*(\mathbf{Q}\mathrm{Div}(X))$. Similarly, we may write every $Z' \in \mathbf{Q}ZX'(K_a)$ as $Z'_1 + Z'_2$, where $Z'_1 \in \mathbf{Q}Z_0X'(K_a)$ and $Z'_2 = \phi^*(Z_2) \in \phi^*(\mathbf{Q}ZX(K_a))$. By bilinearity and the fact that Green's pairings extend the Néron pairing, any g' satisfying (1) and (2) must be given, in the above notation, by

$$g'_v(D', Z') = \langle D'_1, Z_1 \rangle_v + g_v(D_2, \phi_* Z') + g_v(\phi_* D'_1, Z_2). \tag{4}$$

This shows uniqueness. For existence, define g' by (4). To see that this is well defined, observe that any two decompositions $D = D_1 + D_2$ as above differ by the pullback of a $\mathbf{Q}$-divisor of degree 0 on X, and similarly for Z; now apply the functoriality of the Néron pairing and the fact that g extends that pairing. Property (1) now follows by setting $D'_1 = 0$ in (4). Setting $Z'_1 = 0$, $Z = Z_2$ in (4) and then applying (1) to $g_v(D_2, \phi^* Z)$ yields (2).

(ii) This follows from (1) by putting $Z' = \phi^* Z$. ∎

Note that the above proposition and the construction of g' yield formulas relating the transition function τ between two Green's pairings g, γ, on X and the transition function τ' between their pullbacks $g' = \phi^* g$, $\gamma' = \phi^* \gamma$ to X'. Namely, $\tau(Z) = \tau'(\phi_* Z)$ and $\tau'(Z') = (1/\deg \phi)\, \tau(\phi_* Z')$.

Note also that the uniqueness of $g' = \phi^* g$ in 2.4.3 implies functoriality. Namely, if $\phi\colon X' \to X$ and $\sigma\colon X'' \to X$ then $(\phi\sigma)^* g = \sigma^* \phi^* g$.

Next we consider base change. Let K, M be as above, let K' be a finite extension of K, and let M' be a proper set of absolute values of K' such that each element of M' extends an element of M. If X is a curve over K, then there is an induced K'-curve, $X' = X \times_K K'$. Thus we may consider Green's pairings on X over K' with respect to M'. The inclusion $K \hookrightarrow K'$ induces an inclusion $\mathrm{Div}(X) \hookrightarrow \mathrm{Div}(X')$ of divisor groups and an isomorphism $ZX'(K_a) \xrightarrow{\sim} ZX(K_a)$ between the groups of geometric zero cycles. In each case we will identify the groups with their images. We then have:

2.4.4, Proposition. *Let K, M be as in 2.4.1, let X be a K-curve, and let g be a Green's pairing on X with respect to M.*

(a) *Let K' be a finite extension of K and M' a proper set of absolute values on K' lying over the elements of M. On the K'-curve $X' = X \times_K K'$, there is a unique Green's pairing g' such that $g'_{v'}$ extends g_v for all $v' \in M'$ with $v' \mid v \in M$. Moreover, $g'_{v'} = g'_{v''}$ if $v', v'' \mid v$.*

(b) *There is a unique symmetric bilinear map $g_a\colon \mathbf{Q}ZX(K_a) \times \mathbf{Q}ZX(K_a) - \Delta \to \mathbf{R}^M$ which extends g, and which satisfies*

$$g_{a,v}(D, Z) = g'_{v'}(D, Z) \tag{*}$$

for every choice of K', M' as in (a), *for all $v' \in M'$ with $v' \mid v \in M$, and for all $(D, Z) \in \mathbf{Q}\mathrm{Div}(X') \times \mathbf{Q}ZX(K_a) - \Delta$.*

Proof. (a) After shinking M, we may assume that the contraction map $\pi: M' \to M$ is surjective. Let σ be a section of π. Choose any Green's pairing $\gamma': \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta \to \mathbf{R}^{M'}$ on X' over M', and let $\gamma: \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta \to \mathbf{R}^M$ be the composition of γ' with the map $\sigma^*: \mathbf{R}^{M'} \to \mathbf{R}^M$ induced by σ. Then γ is a Green's pairing on X with respect to M, and so there is a locally bounded continuous transition function $\tau: \mathbf{Q}ZX(K_a) \to \mathbf{R}^M$ between g and γ. Define the Green's pairing g' on the K'-curve X' over M' by $g'_{v'}(D, Z) = \gamma_{v'}(D, Z) - (\deg Z)\, \tau_{\pi(v')}(D) - (\deg D)\, \tau_{\pi(v')}(Z)$. Thus the transition function τ' between g' and γ' is the composition of τ with σ_*, and it is immediate that g' extends g.

To show uniqueness, we may assume that M consists of a single place v and that M' consists of a single place v'. Suppose that g' and γ' were both extensions of g. Let $\tau': \mathbf{Q}ZX(K_a) \to \mathbf{R}$ be the transition function between g' and γ'. Every $(D, Z) \in \mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}ZX(K_a) - \Delta$ also lies in the corresponding group for X', so $(\deg Z)\, \tau'(D) + (\deg D)\, \tau'(Z) = \gamma'(D, Z) - g'(D, Z) = 0$. Taking $Z = P - Q$, where P, Q are distinct K_a-points, and taking D to be a divisor of positive degree, we find that τ' is constant on $X(K_a)$. But now taking D to be a divisor of degree 0 and Z to be a K_a-point, we find that this constant must be 0. So $\gamma' = g'$, proving uniqueness.

Because of the uniqueness, $g'_{v'} = g'_{v''}$ if $v', v'' \mid v$ for some $v \in M$. For otherwise, we could define a γ' by interchanging $g'_{v'}$ and $g'_{v''}$, thus violating uniqueness if these two local Green's pairings were unequal.

(b) Given $(Y, Z) \in \mathbf{Q}ZX(K_a) \times \mathbf{Q}ZX(K_a) - \Delta$, let K' be a finite extension of K such that Y is defined over K'; i.e., $Y \in \mathrm{Div}(X')$, viewing $X' = X \times_K K'$ as a K'-curve. Let M' be the set of absolute values of K' extending M, let g' be the extension of g given in (a), and define $g_{a,v}(Y, Z) = g'_{v'}(Y, Z)$, for any $v' \mid v$. This is independent of v' by the last assertion in (a) and is independent of the choice of K' because of the uniqueness assertion in (a). So g_a is well defined. The map g_a is bilinear because each g' is, and g_a is symmetric because g' is on $\mathbf{Q}\mathrm{Div}(X') \times \mathbf{Q}\mathrm{Div}(X') - \Delta$. Finally, (*) holds by construction, and (*) also forces uniqueness, since every element of $\mathbf{Q}ZX(K_a)$ lies in some $\mathbf{Q}\mathrm{Div}(X')$. ∎

While a Green's pairing can be regarded as a pairing into the group of M-constants, we may also add the local contributions and obtain a global real-valued "height pairing" in the case that there is a product formula (e.g., for global fields). Namely, let F be a field with a proper set M_F of absolute values, such that

$$\prod_{v \in M_F} |a|_v = 1$$

for all non-zero $a \in F$. With F fixed, we consider finite extensions K of F.

Each K is thus equipped with a proper set M_K of absolute values $|\ |_v$ extending the places in M_F, and for each $v \in M_K$ we take $\|\ \|_v = |\ |_v^{d(v)}$, where $d(v)$ is the degree of the completion K_v over the corresponding completion of F. Thus K satisfies a product formula

$$1 = \prod_{v \in M_K} \|a\|_v = \prod_{v \in M_K} |a|_v^{d(v)} \qquad (a \in K,\ a \neq 0).$$

In this situation, we define the *degree* of an M-constant c to be

$$\deg c = \sum_{v \in M_K} d(v)\, c_v.$$

Thus the degree of a principal M-constant is 0.

Given a divisor D on a K-variety X, a Weil function ϕ_D for D (over the set M_K) induces a well-defined real-valued function $\phi_D(P) = \sum_{v \in M_K} d(v)\, \phi_D(P, v)$ on $X(K_a) - \operatorname{supp} D$. In particular, let $F(z_0, ... z_n)$ be a homogeneous polynomial, with Green's function $g_F = -\log |F|$ on $X \subset \mathbf{P}^n_K$. Then the induced function ϕ_D: $X(K_a) - \operatorname{supp}(F) \to \mathbf{R}$ is given by $g_F(P) = -\log(\|F\|\ (P))$, where we set

$$\|F\|\ (P) = \|F\|_M\ (P) = \prod_{v \in M_K} \|F\|_v\ (P) = \prod_{v \in M_K} |F|_v\ (P)^{d(v)}.$$

Note that $g_{(f)} = g_{F'}/g_{F''}$ is trivial if $f = F'/F''$ is a rational function, by the product formula. Thus ϕ_D depends only on the linear equivalence class of D.

Observe also that $g_F(P) = -\log \|F\|\ (P)$ is equal to the "metric height" $h_L(P)$ of P in the sense of Vojta [Vo, pp. 66–67], where L is the line bundle $O(\operatorname{div}(F))$. In particular, if F is the linear form z_0 then $L = O(1)$ and

$$g_{z_0}(P) = h_{O(1)}(a_0 : \cdots : a_n) = \sum_{v \in M_k} \log \|(a_0, ..., a_n)\|_v,$$

and which is equivalent to the "naive" projective height (but differs by a bounded amount at the infinite primes).

Similarly, any Green's pairing on a smooth projective K-curve X induces a real-valued "height pairing" $(Y, Z) \mapsto Y \cdot Z = \sum_{v \in M_K} d(v) \langle Y, A \rangle_v$ on $\mathbf{Q}\mathrm{Div}(X) \times \mathbf{Q}\mathrm{Div}(X) - \Delta$. Note that if $f \in K(X)^*$, and if $D \in \mathbf{Q}\mathrm{Div}(X)$ has support disjoint from that of (f), then

$$\begin{aligned}(f) \cdot D &= (\deg g(f))(\deg D) - \log \|(f)\|\ (D) \\ &= (\deg g(f))(\deg D),\end{aligned}$$

since $-\log \|(f)\|\ (D) = -\log \prod (|f(D)|_v)^{d(v)} = 0$ by the product formula (using the fact that $f(D)$ is a well-defined element of K).

2.4.5 THEOREM. *Let X be a smooth projective curve over a field K satisfying a product formula and let g be a Green's pairing on X over K relative to M_K. Then the induced real-valued pairing on* $\mathbf{Q}\mathrm{Div}(X)\times\mathbf{Q}\mathrm{Div}(X)-\Delta$ *has a unique extension to a bilinear real-valued pairing on all of* $\mathbf{Q}\mathrm{Div}(X)\times\mathbf{Q}\mathrm{Div}(X)$ *such that* $(f)\cdot D=(\deg g(f))(\deg D)$ *for all* $f\in K(X)^*$ *and* $D\in\mathbf{Q}\mathrm{Div}(X)$.

Proof. As observed prior to the statement of the proposition, $(f)\cdot D=(\deg g(f))(\deg D)$ provided that (f) and D have disjoint support. Now for an arbitrary pair $(D,D')\in\mathbf{Q}\mathrm{Div}(X)\times\mathbf{Q}\mathrm{Div}(X)$, there exists a non-zero $f\in K(X)$ such that $(D+(f),D')\in\mathbf{Q}\mathrm{Div}(X)\times\mathbf{Q}\mathrm{Div}(X)-\Delta$, and we define $D\cdot D'=(D+(f))\cdot D'$. This pairing is well defined, because of the above observation. Moreover, it is easily seen that this pairing is bilinear, that it extends the original pairing, and that it satisfies the condition on pairing with principal divisors. ▮

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